"Exact numeric solutions of non-linear PDEs: an application of machine efficient Chebyshev methods"

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We extend the work of Brisebarre, Muller, Tisserand, and Chevillard [2, 3] on machine-efficient Chebyshev approximation, so that it is applicable to non-linear PDEs, and in particular we explore the application of this method to the Korteweg-de Vries equation [4, 6].

Current numerical algorithms to solve this differential equation for high accuracy are computationally expensive; and methods that use polynomial approximation such as spectral methods are more efficient in this case. This is because high accuracy addition, subtraction, and multiplication are efficiently implemented in general-purpose processors [2, 5, 7, 9, 11].

The proposed approach adopts the usual Chebyshev method except that coefficients are chosen so that they are efficiently handled by current computer hardware. We implement the machine-efficient Chebyshev approximation as shown in Equation 1.

$$f(x) \approx \sum_{n=0}^{\infty} \frac{a_n}{2^m} x^n \;\text{ where }\; a_0, ..., a_n \;\in \mathbb{Z}, \;\text{ and } m \;\in\; \mathbb{N} \tag{1}$$

That is, the coefficients are coerced to be of the form $\frac{a_n}{2^m}$ with $a_n \in \mathbb{Z}$.

Our practical calculations have been performed using Müller's iRRAM [8] exact arithmetic package. The iRRAM will allow us to have a high bit-accuracy coefficients. Thus, this enables us to evaluate functions accurate to 1,000,000 decimal places. We found that these machine efficient approximations do indeed improve the efficiency with which these operations can be performed [1]. The total error(E_t) of performing the machine-efficient approximation is defined in Equation 2.

$$E_t = \frac{n}{2^m} + \frac{2(b-a)^{n+1}}{4^{n+1}(n+1)!}\; max_{a \leq x \leq b} |f^{n+1}(x)| \tag{2}$$

where a, and b are the end points of the interval [a,b], and n is the degree or the order of Chebyshev approximation. The first term is the additional error for machine efficient coefficients.

The Korteweg-de Vries equation is a non-linear partial differential equation. This equation describes soliton waves [10], and is defined in Equation 3.

$$\frac{\partial u}{\partial t} + cu\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{3}$$

The Korteweg-de Vries equation can be solved by analytical methods or numerical methods. The most common analytical method for solving this equation is the inverse scattering method.

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Chebyshev Order	Decimal Places	Time(seconds)
36	100	540.2745
60	128	3996.09
70	143	7357.08
80	154	12373.6

Table 1: Time Analysis in seconds

The inverse scattering method is computationally expensive. On the other hand, the most common numerical methods to solve this equation are Finite difference schemas and power series. The main cause of errors in Finite difference methods are the choice of the step size and not the use of floating-point arithmetic. Therefore, these techniques is not useful for the application of exact arithmetic where we guarantee the accuracy of the results. The power series approach is much slower than other numerical techniques [4, 10].

Experimental evaluation demonstrates that machine efficient approximations do indeed improve the efficiency with which these operations can be performed. Fast evaluation of functions and solution to Partial Differential Equations can be performed using machine-efficient Chebyshev approximation accurate to 1,000,000 decimal places. Table 1 shows the results of applying this idea to the Korteweg-de Vries equation. Moreover, the proposed technique can be easily extended to higher dimensional problems.

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