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High-Accuracy Machine-Efficient Chebyshev Approximation: an Application to Spectral Methods for Sobolev Spaces

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Abstract

Machine-Efficient Chebyshev Approximation is a technique that permits practical evaluation of transcendental functions within a computable arithmetic, such as the computable reals. The approach adopts the usual Chebyshev method so that coefficients are efficiently handled by current computer hardware. The proposed technique has an application to spectral methods for sobolev spaces. A practical demonstration of this work is presented using Müller's iRRAM exact arithmetic package. Experimental evaluation demonstrates that machine efficient approximations do indeed improve the efficiency with which these operations can be performed.

Keywords: `Chebyshev Polynomial, Computable Functions, Exact Real Arithmetic, Machine-Efficient Approximation, Partial Differential Equations.`

1. Introduction.

Differential equations are important numerical operations for most engineering problems. Therefore, it is crucial to find an efficient and accurate method to solve them. The spectral method is a powerful technique to solve these differential equations. In this paper we use Chebyshev spectral method to provide a highly efficient and accurate approximation of higher-dimensional functions and particular solution of certain partial differential equations.

Current numerical algorithms to solve differential equations are computationally expensive. Methods that use polynomial approximation such as spectral method are more efficient. The main reason behind this is addition, subtraction, and multiplication are efficiently implemented in general-purpose processors [5,11,21,22,24].

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Polynomial approximation such as Chebyshev approximation has coefficients that are represented by a finite number of bits in modern processors. This is due to the limitation of the floating-point arithmetic. However, our practical calculations have been performed using Müller's iRRAM [22] exact arithmetic package. The iRRAM will allow us to have a high accuracy bit's representation for the coefficients. Thus, this enables us to evaluate function accurate to 1,000,000 decimal places.

The method proposed in this paper aims to increase the efficiency with which the operations can be performed through using machine-efficient coefficients. The idea is to make the Chebyshev approximation's coefficients represented in the form: $\frac{N}{2^m}$, where N and m are integers. These machine efficient approximations do indeed improve the efficiency with which mathematical operations can be performed [5,8].

We restrict our work to computable functions, of which there are many definitions [4,17,18,23]. Basically, we look at continuous functions, as discontinuous functions are not computable at their points of discontinuity [15].

The novelty of this work is to exploit the original work of Brisebarre, Muller, Tisserand and Chevillard on machine-efficient Chebyshev approximation [5,8] in the new context of solving PDEs.

In Section 2 and 3 we describe the idea of machine-efficient Chebyshev approximation. In Section 4 we show some examples of its application to spectral methods. In Section 5 we evaluate this technique compared to other numerical techniques. Section 6 is the conclusion.

2. Chebyshev Method.

We choose Chebyshev polynomials as they provide a good polynomial approximation [1,6]. The Chebyshev polynomials are defined recursively by:

$$T_n(x) = cos\ n\theta \quad \text{where } x = cos\theta \tag{1}$$

where x ∈ [-1,1], and $\theta \in [0,\pi]$.

Recurrence relation can also be obtained by:

$$T_n(x) = 2x \cdot T_{n-1}(x) - T_{n-2}(x) \quad n = 2, 3, ... \tag{2}$$

with initial conditions:

$$T_0(x) = 1 \ \ , \ and \ \ T_1(x) = x$$

Chebyshev series expansion of a function $f(x)$ is defined in Equation (3).

$$f(x) \simeq \sum_{i=0}^{\infty}{}' \, c_i T_i(x) \tag{3}$$

where c_i are the coefficients:

$$c_0 = \frac{1}{n+1}\sum_{k=0}^{n} f(x_k)\ T_0(x_k) = \frac{1}{n+1}\sum_{k=0}^{n} f(x_k)$$

$$\begin{aligned} c_j &= \frac{2}{n+1}\sum_{k=0}^{n} f(x_k)\ T_j(x_k) \\ &= \frac{2}{n+1}\sum_{k=0}^{n} f(x_k)\ cos(j\frac{2n+1-2k}{2n+2}\pi) \end{aligned}$$

The nodes or the points x_k used in the coefficients calculations are the Chebyshev nodes. Mason and Handscomb [20] used the following Equation to define the Chebyshev nodes of $T_{n+1}(x)$:

$$x_k = cos(\frac{(k-\frac{1}{2})\pi}{n+1})$$

where $k = 0, ..., n$.

3. Machine-Efficient Chebyshev Approximation

The idea of machine-efficient polynomials was proposed by Brisebarre, Muller, Tisserand and Chevillard [5,8]. They stated that computing systems which use functions in their implementations need polynomial approximations. The best polynomial approximation has coefficients that cannot be represented exactly with a finite number of bits. The approximation uses finite-precision arithmetic; hence the coefficients are usually rounded to the nearest multiple of 2^{-mi} [5,8]. The aim is to find the best truncated polynomial approximation which is not necessarily the best minimax approximation [16,21]. From [5,8] the truncated polynomial is defined by Equation (4).

$$\rho_n^{[m_0,m_1,..,m_n]} = \left\{\frac{a_0}{2^{m_0}} + \frac{a_1}{2^{m_1}}x + ... + \frac{a_n}{2^{m_n}}x^n\right\} \tag{4}$$

where $a_0, ..., a_n \quad \in \mathbb{Z}$.

The way we implement machine-efficient coefficients is quite different from the above. The first step in obtaining a machine-efficient Chebyshev approximation is to find the Chebyshev series that approximates the required functions. The next step is to find a machine-efficient version of that approximation. We implement the machine-efficient coefficients as shown in Equation (5). The main difference is that all m are equal for all terms. Having variables m_i as in Equation (4) yields the best accuracy at reduced cost, since the precision used is the minimum required. However, we choose to have a uniform and maximal accuracy for all coefficients, because since we are using an arbitrary precision framework, we are more concerned with the time complexity: we aim to achieve high accuracy within a reasonable time.

$$f(x) \approx \sum_{n=0}^{\infty} \frac{a_n}{2^m} x^n \tag{5}$$

where $a_0, ..., a_n \in \mathbb{Z}$, and $m \in \mathbb{N}$.

For example, if we choose $m = 10$, then we have the following Equation:

$$\rho_n^{[10]} = \left\{ \frac{a_0}{2^{10}} + \frac{a_1}{2^{10}} x + ... + \frac{a_n}{2^{10}} x^n \right\} \tag{6}$$

where $a_0, ..., a_n \in \mathbb{Z}$.

To find the machine-efficient coefficient a_i we apply rounding to the exact coefficients A_i as follows:

$$a_i = \lfloor 2^m \cdot A_i \rceil \tag{7}$$

This approach may lead to an important loss of accuracy when implemented using floating-point arithmetic. However, the iRRAM exact arithmetic package allows us to have high bit-accuracy coefficients a_i which, if carefully chosen, can overcome this loss of accuracy.

4. Application to Spectral methods for Sobolev Spaces

The machine-efficient Chebyshev method has an application to spectral methods for sobolev spaces. In this section we show the use of this technique to solve PDEs such as the heat equation.

4.1. *One-Dimensional Chebyshev Approximation*

Let $f(x)$ be a regular function to be approximated by Chebyshev polynomial within [-1,1]. This function can be approximated by Chebyshev interpolation of degree N using Equation (8).

$$f(x) \approx \sum_{n=0}^{N} a_n \, T_n(x) \tag{8}$$

where $T_n(x)$ are the Chebyshev polynomial, and a_n are the approximation coefficients.

4.2. *Multi-Dimensional Chebyshev Approximation*

The Chebyshev approximation procedure described in Section 4.1 and in many papers and books [1,16,20,21] can easily be extended to higher dimensional cases [2,3,9,12,25]. For example, the Chebyshev approximation for [-1,1]×[-1,1] takes the form defined in Equation (9).

$$f(x_1, x_2) \approx p_{N_1,N_2}(x_1, x_2) = \sum_{k=0}^{N_1} \sum_{l=0}^{N_2} a_{k,l}\, T_k(x_1)\, T_l(x_2) \tag{9}$$

where N_1 and N_2 are the degree of Chebyshev approximation in the direction of x_1, and x_2. Equation (10) defines the coefficients $a_{k,l}$.

$$a_{k,l} = \frac{4}{N_1 N_2 c_k c_l} \sum_{K=0}^{N_1} \sum_{L=0}^{N_2} \frac{f(x_K, x_L)}{c_K c_L}\, cos(k\frac{2N_1+1-2K}{2N_1+2}\pi)\, cos(l\frac{2N_2+1-2L}{2N_2+2}\pi) \tag{10}$$

where $c_0 = c_N = 2$, and $c_j = 1$ for $1 <= j <= N-1$.

This technique is one case of the spectral method. Spectral methods are distinguished by the choice of trail functions. These trail functions are global smooth functions. The most common choices of trail functions are Fourier series and Chebyshev polynomials [10,7,13,14,19]. We choose here Chebyshev polynomials as they provide a good polynomial approximation [20]. Another important reason is that Fourier series are not always a good choice with non-periodic boundary conditions [10].

4.3. *Numerical Results*

Table 1. Chebyshev Nodes for Time when $N = 5$

Chebyshev Nodes
+0.1000000000000000000000000E+0001
+0.8090169943749474241022934E+0000
+0.3090169943749474241022934E+0000
-0.3090169943749474241022934E+0000
-0.8090169943749474241022934E+0000
-0.1000000000000000000000000E+0001

Numerical tests have been performed to find the solution of heat equation using higher order Chebyshev approximation. These tests do indeed demonstrate the effectiveness of the proposed algorithm. The standard domain rectangle [-1,1]×[-1,1] is used on the linear heat equation that is de-

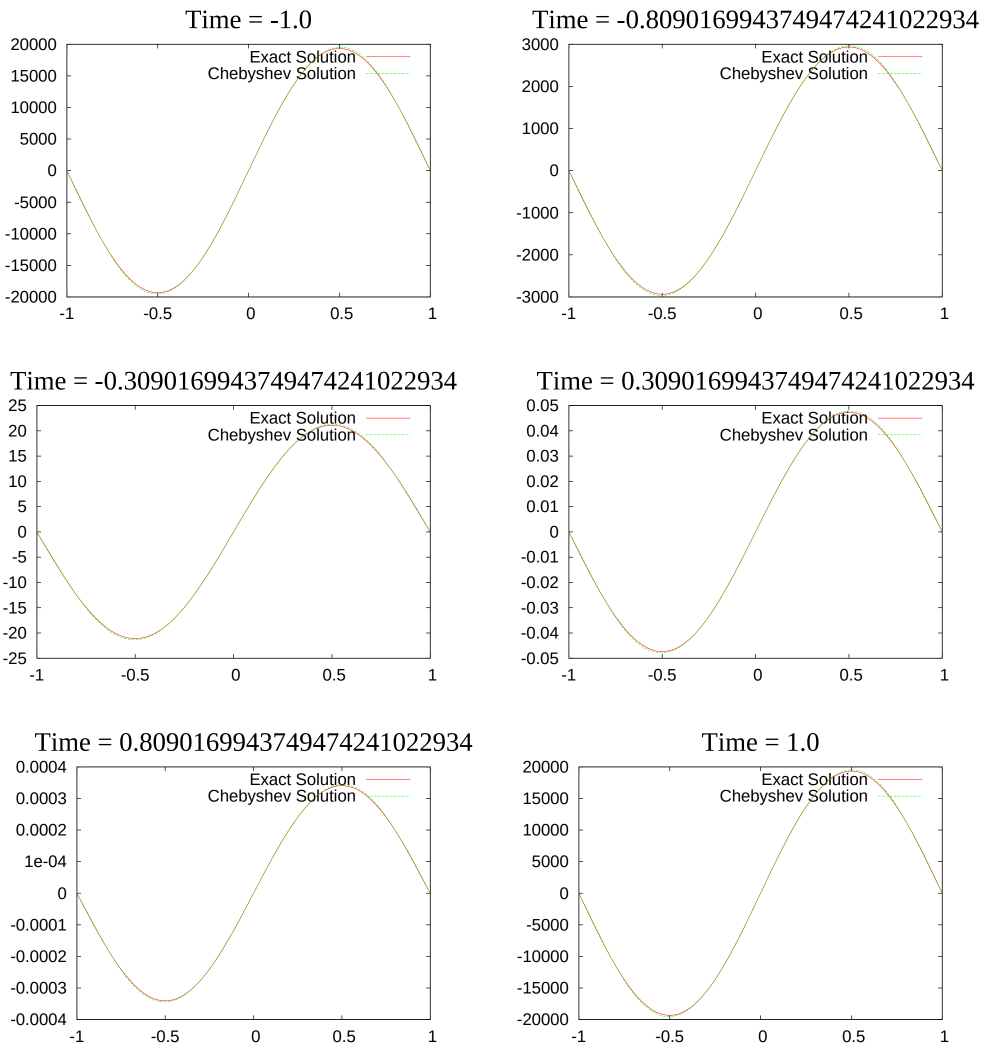

Fig. 1. Chebyshev Approximate Solutions

fined in Equation (11).

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{11}$$

with homogeneous Dirichlet boundary conditions:

$$u(1,t) = 0$$
$$u(-1,t) = 0$$

For the particular periodic initial condition:

$$u(x,0) = sin\pi x \tag{12}$$

the exact answer is:

$$u(x,t) = e^{-\pi^2 t}\ sin\pi x \tag{13}$$

the Chebyshev approximate solution has the following representation:

$$u(x,t) = \sum_{k=0}^{N_1}\sum_{l=0}^{N_2} a_{k,l}\ T_k(x)\ T_l(t) \tag{14}$$

Table 1 shows the Chebyshev nodes or points for the time variable t when $N_2 = 5$. Figure 1 shows the Chebyshev approximate solution to the heat equation defined in Equation (11) at different times where $N_1 = N_2 = 5$.

4.4. *Machine-Efficient Chebyshev Solutions*

This approach may lead to an important loss of accuracy when implemented using floating-point arithmetic. However, the iRRAM exact arithmetic package allows high bit accuracy coefficients a_i that, if carefully chosen, can cancel this lose of accuracy. Figure 2 shows the error between the machine-efficient Chebyshev approximate solutions and the standard Chebyshev approximate solutions for the heat equation described in Section 4.3. The variable m is the machine-efficient denominator power. The error graphs in Figure 2 show that the machine-efficient Chebyshev approximation gets closer to the standard Chebyshev approximation as the value of m increases.

5. Evaluation

5.1. *Finite Difference*

The finite difference method is a numerical method for approximating the solution to differential equations. This method works by first dividing the domain into smaller sub-domains, then local polynomials of low order are used. This means that finite difference method use discrete numerical approximation to the derivative. In other words, the main idea of finite difference is to replace derivatives with linear combinations of discrete function values.

The main two sources of errors in this method are the round-off and truncation errors. The round-off error is caused by computer rounding of

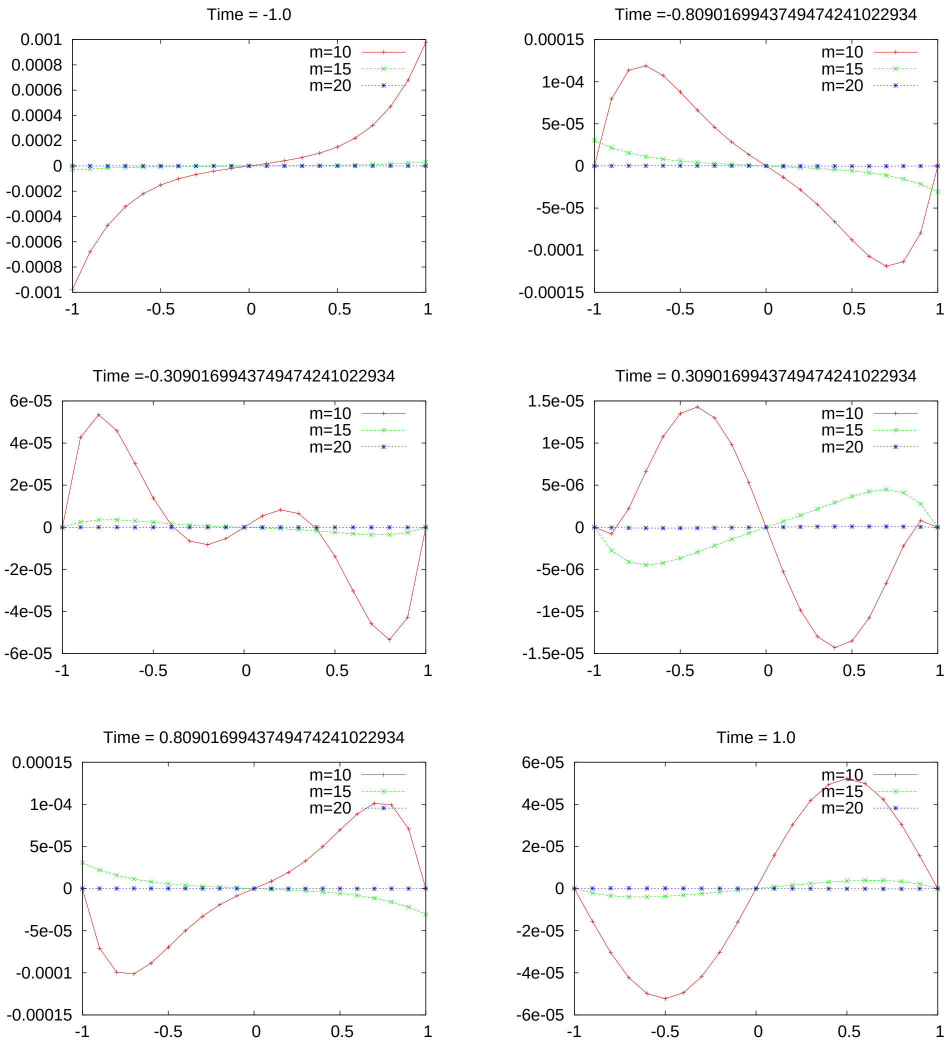

Fig. 2. Machine-Efficient Chebyshev Approximate Solutions Error

decimal numbers, while the truncation error is the difference between the finite difference method and the exact solution of the differential equation assuming no round-off. More accuracy is gained in this method by decreasing the size of the sub-domains [3].

The finite difference method differs from the Chebyshev spectral method by the choice of the trial functions: spectral methods use global smooth functions, while the finite difference method uses overlapping local polynomials of low order [10].

The same example of the heat equation in section 4.3 is used to compare the two methods. Canuto et al [10] compare the results when $t = 1$ for a Chebyshev spectral method and a second-order finite-difference method. Their experiments show the former to be more accurate than the latter. These results are shown in Table 2.

Table 2. Maximum error for Heat Equation

N	Chebyshev Method	Finite-Difference
8	4.58E-4	6.44E-1
10	8.25E-6	3.59E-1
12	1.01E-7	2.50E-1
14	1.10E-9	1.74E-1
16	2.09E-11	1.35E-1

5.2. *Finite Element*

Finite element method is another numerical technique for approximating the solution to partial differential equations and integral equations. This method can be applied to a wide range of physical and engineering problems. It provides more flexibility to solve complex models.

In finite element method, the domain is divided to sub-domains, then local polynomials or functions of fixed degree are used on the sub-domains. Finite element method works in a similar way to the spectral method, the main difference being that the spectral method approximates the solutions using a combination of continuous functions (Chebyshev polynomials), while finite element method approximates the solution using a combination of piece-wise continuous functions that are non-zero on the sub-domains.

Finite element method differ from finite difference method in some points. For example, finite element method has the ability to handle complicated geometric. On the other hand, The finite difference method is easier to implement.

Finite element is a local approach, while the spectral method is a global one, hence the spectral method provides more accurate approximation when the solution is smooth.

6. Conclusion

This research is focused in solving the accuracy problem in the computer arithmetic for numerical analysis. The primary aim of this research is to implement high-accuracy solutions to partial differential equations.

The main aims is to increase the accuracy and improve the performance of the numerical analysis problems as they are needed in most of the engineering applications. Our goal is to devise a sound solution to the problem that is efficient in both time and accuracy.

In this paper we showed that the solution to the two dimensional heat equation can be achieved using higher dimensional machine-efficient Chebyshev approximation. The proposed technique can be easily extended to more than two dimensional problems. Practical experiments proved that these machine-efficient solutions do indeed improve the performance with which these operations can be performed. Fast evaluation of functions and solution to Partial Differential Equations can be performed using machine-efficient Chebyshev approximation. Moreover, the proposed technique provides accurate solutions to one million decimal places.

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