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Abstract

In [Russell 1903, Russell 1906] Bertrand Russell mentioned the following possible definitions of conjunction, disjunction, negation and existential quantification in terms of implication and universal quantification, exploiting impredicative universal quantifiers over all propositions. In [Prawitz 1965] Dag Prawitz showed that these definitions hold in intuitionistic second order logic. More recently, these definitions have been used to represent logic in various impredicative type theories. this treatment of logic is distinct from the more standard Curry-Howard representation of logic in a dependent type theory.

The main aim of this paper is to compare, in a purely logical, non-type-theoretic setting, this Russell-Prawitz representation of intuitionistic logic with other possible representations. It turns out that associated with the Russell-Prawitz representation is a lax modal operator, which we call the Russell-Prawitz modality, and that any lax modal operator can be used to give a translation of intuitionistic logic into itself that generalises both the double negation interpretation, double negation being a paradigm example of a lax modality, and the Russell-Prawitz representation.

1 Introduction

In [Russell 1903, Russell 1906] Bertrand Russell mentioned the following possible definitions of conjunction, disjunction, negation and existential quantification in terms of implication and universal quantification.

$$\begin{aligned}
[\phi_1 \wedge \phi_2] &\equiv \forall p[(\phi_1 \to (\phi_2 \to p)) \to p] \\
[\phi_1 \vee \phi_2] &\equiv \forall p[(\phi_1 \to p) \to ((\phi_2 \to p) \to p)] \\
[\neg\phi] &\equiv \forall p[\phi \to p] \\
[\exists x \phi(x)] &\equiv \forall p[\forall x(\phi(x) \to p) \to p]
\end{aligned}$$

In these definitions the variable p is intended to range over all propositions.

In [Prawitz 1965] Dag Prawitz showed that the above equivalences can be proved in second order intuitionistic logic, and even in ramified second order logic, provided that, in each case the level of the bound proposition variable is suitably chosen.

In fact it is also easy to see that the above equivalences can be used as definitions in the $\to$, $\forall$ fragment of second order intuitionistic logic, thereby reducing full second order intuitionistic logic to this fragment. But the argument does not carry over to ramified second order logic.

The idea is also used to express logic in Girard's system F, [Girard 1971, Girard et al. 1990], and is the standard approach to representing logic in the calculus of constructions, [Coquand 1990], and its extensions. In particular the standard approach to represent logic in the type theory of Lego, [Luo & Pollack 1992, Luo 1994], and also sometimes the type theory of Coq, [Barras et al. 1996], is to use the above Russell-Prawitz representation, where the variable p ranges over the impredicative type called $Prop$ in Lego.

In this representation propositions are represented as objects of type $Prop$. These objects are themselves types (or are names of types in some treatments of the calculus of constructions), and the logical operations are represented as operations on this type. The central rule for the type $Prop$, which gives it its impredicative power, is the rule that gives

$$(\Pi x : A)B(x) : Prop$$

for any type A and any family $B(x) : Prop$ for $x : A$. Writing $(\forall x : A)B(x)$ for $(\Pi x : A)B(x)$ and $A \to B$ for $(\Pi \_ : A)B$ [1] following the Curry-Howard correspondence, we get universal quantification, over any type A, and implication as logical operations on $Prop$.

The Lego and Coq systems implement versions of the calculus of constructions that also include the type forming operations of Martin-Löf's type theory such as strong Σ types, other forms of inductive types and predicative type universes. The standard approach to representing logic in Martin-Löf's type theory, [Martin-Löf 1984] is to represent propositions as types in general, not only as types in Prop, and use the Curry-Howard representation for all the logical operations.

Thus, for the Lego and the Coq type theories we have two distinctive ways to represent logic, the one that has here been called the Russell-Prawitz representation, where propositions are represented as types in the type $Prop$, and the Curry-Howard representation where propositions are represented as arbitrary types. What is the relationship between these two ways of representing logic? The aim of the paper will be to explore this relationship in a purely logical setting. It will be convenient to focus on standard theories extending many-sorted minimal logic, allowing one of the sorts to be a sort π of propositions. There should be no difficulty in carrying over the discussion to other contexts like type theory itself, but we leave that for another paper. The main contributions of this paper are to highlight the notion of a lax modality, $\neg\neg$ being the paradigm example, to generalise the double negation interpretation to any lax modality and to characterize the Russell-Prawitz reinterpretation of intuitionistic logic as a variant of this generalisation for a lax modality P that we choose to call the Russell-Prawitz modality. This notion of a lax modality is not an unfamiliar one, particularly in the context of locale (or cHa) theory, where it was been called a J-operator in [Fourman & Scott 1979] and a nucleus in [Simmons 1978]. See also [Johnstone 1982] and section 2 for some more background.

After we have made precise in section 1 what we will mean by a standard theory our first aim in section 2 will be to formulate the general notion of a lax modality for a standard theory and give many examples. Given any lax modality J we define the J-translation in section 3. In section 4 we consider the 'A'-translation as a variant of the J-translation for one of the simplest kinds of lax modality J. Section 5 is concerned with the double negation translation and their variants. In section 6 we at last introduce the lax modality P and see how the Russell-Prawitz definitions give a variant of the P-translation.

2 Standard Theories

We assume that $\mathcal{L}$ is a language for many sorted intuitionistic predicate logic, having as primitive logical constants the connectives $\perp$, $\to$, $\wedge$, $\vee$ and the quantifiers $\forall x : \sigma$ and $\exists x : \sigma$ for each sort σ. The terms and formulae of $\mathcal{L}$ are defined in a standard way. We write $\mathcal{L}^-$ for the $\perp, \to, \forall$ fragment of the language $\mathcal{L}$.

[1] Here the underscore _ indicates a dummy variable that does not occur in B.

We assume given a standard axiomatisation of intuitionistic logic for $\mathcal{L}$. To be specific we axiomatize it using the rules of inference (MP) $\phi, (\phi \to \psi)/\ \psi$, (UG) $\phi/\ (\forall x:\sigma)\phi$ for each sort σ and the axiom schemes (K) $\phi \to \psi \to \phi$, (S) $(\phi \to \psi \to \theta) \to (\phi \to \psi) \to \phi \to \theta$, $(\forall E)$ $(\forall x:\sigma)\phi \to \phi[t/x]$ if t is a term of sort σ that is free for x in ϕ, $(\forall I)$ $(\forall x:\sigma(\theta \to \phi) \to \theta \to (\forall x:\sigma)\phi$ if x is not free in θ, $(\perp)$ $\perp \to \theta$, $(\wedge)$ $\phi \to \psi \to (\phi \wedge \psi)$, $(\wedge El)$ $(\phi \wedge \psi) \to \phi$, $(\wedge Er)$ $(\phi \wedge \psi) \to \psi$, $(\vee Il)$ $\phi \to (\phi \vee \psi)$, $(\vee Ir)$ $\psi \to (\phi \vee \psi)$, $(\vee E)$ $(\phi \to \theta) \to (\psi \to \theta) \to (\phi \vee \psi) \to \theta$, $(\exists E)$ $\phi[t/x] \to (\exists x:\sigma)\phi$ if t is a term of sort σ that is free for x in ϕ, $(\exists I)$ $(\forall x:\sigma)(\phi \to \theta) \to (\exists x:\sigma)\phi \to \theta$ if x is not free in θ.

The logic ${}^mL$ is L with the axiom scheme $(\perp)$ left out and the logics L^- and ${}^mL^-$ are obtained from L and ${}^mL$ respectively by leaving out all schemes involving the logical operations $\wedge$, $\vee$, $\exists$ that are not in $\mathcal{L}^-$. Thus ${}^mL$ is minimal logic and L^- and ${}^mL^-$ are the fragments of intuitionistic logic and minimal logic that are in the language $\mathcal{L}^-$.

Finally L^c is L (or essentially equivalently ${}^mL^-$) with classical logic; i.e. the scheme $\neg\neg\phi \to \phi$ is added to L, where $\neg\phi$ is defined to be $\phi \to \perp$ as usual.

By a **standard theory** for $\mathcal{L}$ ($\mathcal{L}^-$) we will mean a theory obtained from ${}^mL$ (${}^mL^-$) by adding a set of formulae as additional axioms. We will identify the standard theory with this set of additional axioms. For example the standard theories L and L^c have the additional axioms $\perp \to \theta$ and $\neg\neg\theta \to \theta$ respectively for all formulae θ. If T is a standard theory we will write $T \vdash \phi$ or $\vdash_T \phi$ if ϕ is a theorem of T and write $\phi \ \dashv\vdash_T\ \psi$ if both $T \vdash \phi \to \psi$ and $T \vdash \psi \to \phi$. Also we will write $T + \Gamma$ for the standard theory obtained from T by adding the set Γ of formulae as extra axioms and often write $\Gamma \vdash_T \phi$ for $T + \Gamma \vdash \phi$.

3 Lax Modalities

Let T be a standard theory. A lax modality of T is a certain kind of unary connective defined in T. The terminology *lax modality* comes from Lax Logic, a somewhat peculiar modal logic introduced in [Fairtlough and Mendler 1997]. Essentially that paper is a study of the logic obtained by adding to intuitionistic propositional logic a primitive lax modality, there written O. The logic was motivated by applications to hardware verification, where the modality is used to formalise a notion of correctness up to constraints. The same logic, but now called CL-logic, was also introduced in [Benton, Bierman and de Paiva 1998] and was motivated by the quite different considerations coming from Moggi's computational lambda calculus,[Moggi 1989, Moggi 1991].

We first wish to formulate what we mean by a defined unary connective J of a language $\mathcal{L}$. To do so let $\mathcal{L}(*)$ be obtained from $\mathcal{L}$ by adding a new atomic sentence $*$. Let J be any formula of $\mathcal{L}(*)$. Then for any formula ϕ of $\mathcal{L}$ we let $\mathsf{J}\phi$ be the formula of $\mathcal{L}$ obtained from J by replacing every occurrence of $*$ by ϕ, relabelling any bound variables of J so as to avoid binding any free variables of ϕ. In the obvious way we can also define the notion of a defined unary connective of $\mathcal{L}^-$.

Let T be a standard theory in the language $\mathcal{L}$ ($\mathcal{L}^-$). A defined unary connective J of $\mathcal{L}$ ($\mathcal{L}^-$) is called a **lax modality** of T if the following formulae are theorems of T for all formulae ϕ, ϕ' of $\mathcal{L}$ ($\mathcal{L}^-$).

(J1) $\phi \to \mathsf{J}\phi$,

(J2) $(\phi \to \mathsf{J}\phi') \to (\mathsf{J}\phi \to \mathsf{J}\phi')$.

In this definition the scheme (J2) can be replaced by the combination of the following two schemes.

($\mathsf{J2}_1$) $\mathsf{JJ}\phi \to \mathsf{J}\phi$,

($\mathsf{J2}_2$) $(\phi \to \phi') \to (\mathsf{J}\phi \to \mathsf{J}\phi')$.

Moreover for any lax modality J the following scheme can be derived.

(J3) $\mathsf{J}(\phi \wedge \phi') \leftrightarrow (\mathsf{J}\phi \wedge \mathsf{J}\phi')$.

If also $T \vdash \neg\mathsf{J}\bot$ then we call J a **strict lax modality** of T.

This notion of lax modality is not at all new. In the context of topos theory a lax modality in the local set theory of a topos has been called a 'modality' of the local set theory in [Bell 1988] and is used to generalise the notion of a Grothendieck topology for presheaf toposes to all elementary toposes. Another concept from category theory that is closely related to the notion of a lax modality is that of a strong monad on a category. In particular a strong monad on a cartesian closed category is a lax modality in the $\wedge, \to$ fragment of intuitionistic propositional logic where $\wedge$ and $\to$ are interpreted as the binary cartesian product and exponentiation operations on the objects of the category. In the logic of a locale[2] a lax modality is what has been called a nucleus. See, for example, II 2.2. of [Johnstone 1982].

3.1 Examples

There are many examples of lax modalities. Let us start with the trivial ones $\mathsf{J}_{id} = *$ and $\mathsf{J}_{tr} = (* \to *)$. So $\mathsf{J}_{id}\phi \;\dashv\vdash_T\; \phi$ and $\vdash_T \mathsf{J}_{tr}\phi$ for all formulae ϕ. More interesting examples are obtained as follows. Given any formula α let $\mathsf{J}^\alpha = (\alpha \to *)$, $\mathsf{B}_\alpha = ((* \to \alpha) \to \alpha)$, and $\mathsf{J}_\alpha = (\alpha \vee *)$. The special case $\mathsf{B}_\bot$; i.e. $\neg\neg*$ is a particularly important one and we shall use the more familiar notation $\neg\neg$ and call it the **double negation modality**. New lax modalities can be obtained from old ones in various ways. For example if J and J' are lax modalities then so is $(\mathsf{J} \wedge \mathsf{J}')$. Also, if $\mathsf{J}(x)$ is a lax modality that may have free occurrences of the variable x of sort σ then $(\forall x : \sigma)\mathsf{J}(x)$ is also a lax modality. In general we may compose unary defined connectives J, J' to obtain a defined unary connective JJ'. The composition of two lax modalities need not always be a lax modality. But we do have the following result. If J is a lax modality then so is $\mathsf{J}^\alpha\mathsf{J}$ and also JJ_α. The notion of a lax modality is not of great interest when the logic is classical. In fact for any lax modality J of a classical theory every formula $\mathsf{J}\phi \leftrightarrow (\mathsf{J}\bot \vee \phi)$ is a theorem, so that J is extensionally the same as J_α where α is $\mathsf{J}\bot$, and becomes extensionally the trivial modality J_{id} if J is strict. In the context of cHa-theory these examples of lax modalities, there called J-operators, can be found in [Fourman & Scott 1979], along with formulae relating ways to combine them.

3.2 Some properties of a lax modality

Let J be a lax modality for a standard theory T. We state some easily proved properties of J.

[2] Also called complete Heyting algebra or frame.

Proposition 1 *The following are theorems of T for all formulae ϕ, ψ.*

1. $\mathsf{JJ}\phi \to \mathsf{J}\phi$
2. $\mathsf{J}(\phi \to \psi) \to \mathsf{J}\phi \to \mathsf{J}\psi$
3. $\mathsf{J}(\phi \to \mathsf{J}\psi) \to \phi \to \mathsf{J}\psi$
4. $\mathsf{J}(\forall x : \sigma)\mathsf{J}\psi \to (\forall x : \sigma)\mathsf{J}\psi$

Corollary 2 *If θ is a formula having one of the forms $\mathsf{J}\psi$, $\theta \to \mathsf{J}\psi$, $(\forall x : \sigma)\mathsf{J}\psi$ then*

$$\mathsf{J}\theta \dashv\vdash_T \theta.$$

Proposition 3 *If T is a standard theory for $\mathcal{L}$ then $\mathsf{J}(\phi \wedge \psi) \dashv\vdash_T (\mathsf{J}\phi \wedge \mathsf{J}\psi)$.*

3.3 Relating lax modalities

Let J, J' be lax modalities of a standard theory T for $\mathcal{L}$ ($\mathcal{L}^-$). We define $\mathsf{J} \leq_T \mathsf{J}'$ if $\vdash_T (\mathsf{J}\phi \to \mathsf{J}'\phi)$ for all formulae ϕ of $\mathcal{L}$ ($\mathcal{L}^-$), and if both $\mathsf{J} \leq_T \mathsf{J}'$ and $\mathsf{J}' \leq_T \mathsf{J}$ then we write $\mathsf{J} \equiv_T \mathsf{J}'$.

Proposition 4 *Let J be a lax modality for a standard theory T. Then $\mathsf{J}_{id} \leq_T \mathsf{J} \leq_T \mathsf{J}_{tr}$ and*

$$\mathsf{J} \leq_T \neg\neg \quad \textit{iff} \quad \mathsf{J} \textit{ is a strict lax modality.}$$

Proof. The first part is trivial. For the second part if $\mathsf{J} \leq_T \neg\neg$ then, working informally in T, $\mathsf{J}\bot \to \neg\neg\bot$ so that $\neg\mathsf{J}\bot$. Conversely, assuming that (i) $\neg\mathsf{J}\bot$ we show that $\mathsf{J}\phi \to \neg\neg\phi$. So assume (ii) $\mathsf{J}\phi$ and (iii) $\neg\phi$ to get a contradiction. By (iii) $\phi \to \bot$ and hence $\mathsf{J}\phi \to \mathsf{J}\bot$. So, by (ii), $\mathsf{J}\bot$ contradicting (i). □

4 The J-translation

Any lax modality J of a standard theory T for $\mathcal{L}$ can be used to define a translation of $\mathcal{L}$ into itself. This translation generalises the familiar double negation translation, which has been used to represent classical logic in intuitionistic logic. There is also a version of the translation when T is a standard theory in $\mathcal{L}^-$, but we do not spell it out.

Let J be a lax modality of a standard theory T for $\mathcal{L}$. We define a translation $()^{\mathsf{J}}$ of $\mathcal{L}$ into itself called the J-translation. First we associate with each logical constant of $\mathcal{L}$ its J version. The J version of $\bot$ is $\mathsf{J}\bot$. The J version of $\to$ is the binary operation $\to^{\mathsf{J}}$ on formulae of L given by

$$\phi_1 \to^{\mathsf{J}} \phi_2 \equiv \mathsf{J}(\phi_1 \to \phi_2)$$

for all formulae ϕ_1, ϕ_2 of $\mathcal{L}$. The J versions of $\wedge$ and $\vee$ are defined similarly if they are in $\mathcal{L}$. Also if Q is one of $\forall$, $\exists$ then the J version of Q is the unary operation Q^{J} on formulae of $\mathcal{L}$ given by

$$(Q^{\mathsf{J}}x : \sigma)\phi \equiv \mathsf{J}(Qx : \sigma)\phi$$

for all formulae ϕ of $\mathcal{L}$.

Now ϕ^{J} is defined by structural recursion on ϕ using the following table, where θ is any atomic formula of $\mathcal{L}$, $\circ$ is any binary connective and Q is any of the two quantifiers.

ϕ	ϕ^{J}
θ	$\mathsf{J}\theta$
$\bot$	$\mathsf{J}\bot$
$(\phi_1 \circ \phi_2)$	$(\phi_1 \circ^{\mathsf{J}} \phi_2)$
$(Qx : \sigma)\phi_0$	$(Q^{\mathsf{J}}x : \sigma)\phi_0$

We state some easily proved properties of the J-translation.

Proposition 5 *Let J be a lax modality for a standard theory T for $\mathcal{L}$ or $\mathcal{L}^-$. Then*

1. *(a)* $\mathsf{J}\phi^{\mathsf{J}} \;\vdash\!\dashv_T\; \phi^{\mathsf{J}}$,
   (b) $(\phi \to \psi)^{\mathsf{J}} \;\vdash\!\dashv_T\; \phi^{\mathsf{J}} \to \psi^{\mathsf{J}}$,
   (c) $((\forall x : \sigma)\phi)^{\mathsf{J}} \;\vdash\!\dashv_T\; (\forall x : \sigma)\phi^{\mathsf{J}}$.
2. *Moreover if T is a standard theory for $\mathcal{L}$ then*
   (a) $(\phi \wedge \psi)^{\mathsf{J}} \;\vdash\!\dashv_T\; \phi^{\mathsf{J}} \wedge \psi^{\mathsf{J}}$,
   (b) $((\phi \vee \psi) \to \theta)^{\mathsf{J}} \;\vdash\!\dashv_T\; ((\phi^{\mathsf{J}} \vee \psi^{\mathsf{J}}) \to \theta^{\mathsf{J}})$,
   (c) $((\exists x : \sigma)\phi \to \theta)^{\mathsf{J}} \;\vdash\!\dashv_T\; (\exists x : \sigma)\phi^{\mathsf{J}} \to \theta^{\mathsf{J}}$.

Note that in defining the $()^{\mathsf{J}}$-translation the J version of each logical constant was always used. In fact, by this proposition the ordinary version can be used for $\wedge$, $\to$ and $\forall$ and also for $\bot$ when J is strict. The resulting formula will be equivalent in T to ϕ^{J}.

Definition 6 *If J is a lax modality for a standard theory T the theory T is called* **$(\ )^{\mathsf{J}}$-closed** *if $\vdash_T \phi$ implies $\vdash_T \phi^{\mathsf{J}}$ for all formulae ϕ. If T is a standard theory that is $(\ )^{\mathsf{J}}$-closed for each of its lax modalities J then we will call T a* **lax-closed** *standard theory.*

The following theorem has a straightforward proof that generalises the familiar proof in the special case for L when J is $\neg\neg$.

Theorem 7 *Each of the logics L, ${}^mL$, L^-, ${}^mL^-$ and L^c is lax-closed.*

There are other familiar examples of lax closed theories. For example the theory HA of Heyting Arithmetic is an example of a standard theory for the familiar single-sorted language of Formal Arithmetic; i.e. the language having the binary relation symbol '=' the constant '0', unary function symbol 'S' and the binary function symbols '$+, .$'. It is straightforward to show the following result.

Proposition 8 *HA is lax-closed.*

If Γ is a set of formulae of $\mathcal{L}$ ($\mathcal{L}^-$) let $\Gamma^{\mathsf{J}} = \{\phi^{\mathsf{J}} \mid \phi \in \Gamma\}$. Also let $\Delta_{\mathsf{J}} = \{\mathsf{J}\phi \to \phi \mid \phi$ a formula of $\mathcal{L}(\mathcal{L}^-)\}$.

Theorem 9 *Let J be a lax modality of a standard theory T that is J-closed. Then*

1. *$\Gamma \vdash_T \phi$ implies $\Gamma^{\mathsf{J}} \vdash_T \phi^{\mathsf{J}}$.*
2. *If $T \vdash (\neg\neg\phi \to \phi)^{\mathsf{J}}$ for all formulae ϕ then $\Gamma \vdash_{T^c} \phi$ implies $\Gamma^{\mathsf{J}} \vdash_T \phi^{\mathsf{J}}$.*
3. *If $T \vdash (\mathsf{J}\phi \to \phi)^{\mathsf{J}}$ for all formulae ϕ then $\Gamma \vdash_{T+\Delta_{\mathsf{J}}} \phi$ iff $\Gamma^{\mathsf{J}} \vdash_T \phi^{\mathsf{J}}$.*

5 The 'A'-translation

The simplest non-trivial examples of a lax modality are those of the form J_α; i.e. $(\alpha \vee *)$, for any formulae α. The J_α-translation is essentially the same as the 'A'-translation, which we will here call the α-translation. For any formula ϕ the formula ϕ^α is obtained from ϕ by replacing every atomic formula θ by $(\alpha \vee \theta)$ and replacing $\perp$ by α. In the case when α is $\perp$ this was first used in [Prawitz and Malmnäs 1968] to interpret intuitionistic logic in minimal logic. More recently the general α-translation has been introduced by Friedman, [Friedman 1978] (and also by Dragalin) as a useful tool in connection with proving closure under Markov's rule. See also [Leivant 1985, Murthy 1991]. We have the following connection with the J_α-translation.

Proposition 10 *If T is a standard theory for $\mathcal{L}$ and α is any formula such that $\vdash_T (\perp \to \alpha)$ then $\phi^\alpha \;\dashv\vdash_T\; \phi^{\mathsf{J}_\alpha}$.*

6 The double negation translation

Note that $T + \Delta_{\neg\neg}$ is the classical version T^c of T and, as $\vdash_T (\neg\neg\phi \to \phi)^{\neg\neg}$, part 3 of Theorem 9 gives the familiar result:

Proposition 11 *If T is a $(\ )^{\neg\neg}$-closed standard theory then $\Gamma \vdash_{T^c} \phi$ iff $\Gamma^{\neg\neg} \vdash_T \phi^{\neg\neg}$.*

Part 2 of Theorem 9 can be applied in the case when J is B_α to get the following result.

Proposition 12 *If T is a $(\ )^{\mathsf{B}_\alpha}$-closed standard theory such that $\vdash_T \perp \to \alpha$ then $\Gamma \vdash_{T^c} \phi$ implies $\Gamma^{\mathsf{B}_\alpha} \vdash_T \phi^{\mathsf{B}_\alpha}$.*

Note that the condition $\vdash_T \perp \to \alpha$ holds automatically when T is at least an intuitionistic theory rather than a theory of minimal logic. The key observation here is that

$$(\neg\neg\phi)^{\mathsf{B}_\alpha} \;\dashv\vdash_T\; ((\perp \to \alpha) \to \phi^{\mathsf{B}_\alpha}).$$

6.1 The $(\ )^N$ variant

This variant of the $\neg\neg$-translation is defined as follows. For each formula ϕ of $\mathcal{L}$ the formula ϕ^N of $\mathcal{L}^-$ is defined by structural recursion on ϕ using the following table.

ϕ	ϕ^N
θ	$\neg\neg\theta$
$\perp$	$\perp$
$(\phi_1 \to \phi_2)$	$(\phi_1^N \to \phi_2^N)$
$(\phi_1 \wedge \phi_2)$	$\neg(\phi_1^N \to \neg\phi_2^N)$
$(\phi_1 \vee \phi_2)$	$\neg\phi_1^N \to \neg\neg\phi_2^N$
$(\forall x : \sigma)\phi_0$	$(\forall x : \sigma)\phi_0^N$
$(\exists x : \sigma)\phi_0$	$\neg(\forall x : \sigma)\neg\phi_0^N$

Proposition 13 *1. $\phi^{\neg\neg} \;\dashv\vdash_{^mL}\; \phi^N$ for all formulae ϕ,*

2. If T is a $(\ )^{\neg\neg}$-closed standard theory of $\mathcal{L}^-$, such as $^mL^-$ then, for all formulae ϕ, $\Gamma \vdash_{T^c} \phi$ iff $\Gamma^N \vdash_T \phi^N$.

The definition of ϕ^N can be generalised to some extent by replacing $\perp$ by an arbitrary formula α of $\mathcal{L}^-$. For each formula ϕ of $\mathcal{L}$ the formula $\phi^{\alpha\alpha}$ of $\mathcal{L}^-$ is defined by structural recursion on ϕ using the following table.

ϕ	$\phi^{\alpha\alpha}$
θ	$(\theta \to \alpha) \to \alpha$
$\perp$	α
$(\phi_1 \to \phi_2)$	$(\phi_1^{\alpha\alpha} \to \phi_2^{\alpha\alpha})$
$(\phi_1 \wedge \phi_2)$	$(\phi_1^{\alpha\alpha} \to \phi_2^{\alpha\alpha} \to \alpha) \to \alpha$
$(\phi_1 \vee \phi_2)$	$(\phi_1^{\alpha\alpha} \to \alpha) \to (\phi_2^{\alpha\alpha} \to \alpha) \to \alpha$
$(\forall x : \sigma)\phi_0$	$(\forall x : \sigma)\phi_0^{\alpha\alpha}$
$(\exists x : \sigma)\phi_0$	$((\forall x : \sigma)(\phi_0^{\alpha\alpha} \to \alpha)) \to \alpha$

The next two propositions have easy direct proofs.

Proposition 14 *If α is a formula of $\mathcal{L}^-$ then $\Gamma \vdash_{L^c} \phi$ implies $\Gamma^{\alpha\alpha} \vdash_{{}^mL^-} \phi^{\alpha\alpha}$.*

Proposition 15 *If α is a formula of $\mathcal{L}$ then for all formulae ϕ of $\mathcal{L}$*

1. *If $\vdash_T \perp \to \alpha$ then $\phi^{\alpha\alpha} \dashv\vdash_{{}^mL} \phi^{\mathsf{B}_\alpha}$,*
2. *$\phi^{\alpha\alpha} \dashv\vdash_{{}^mL} (\phi^{\neg\neg})^\alpha$.*

7 The Russell-Prawitz translation

Given a language $\mathcal{L}$ we let $\mathcal{L}_\pi$ be the language obtained from $\mathcal{L}$ by adding a new sort π and a new unary relation symbol true taking one argument of sort π. We will use $p, q, r, etc\ldots$ as variables of sort π and these will be the only terms of sort π. For each π-variable p the atomic formula $\mathsf{true}(p)$ will be abbreviated to just p and the sorted quantifiers $\forall p : \pi$ and $\exists p : \pi$ will be abbreviated to just $\forall p$ and $\exists p$ respectively.

For any set $\mathcal{F}$ of formulae of $\mathcal{L}_\pi$ let $(\forall\mathcal{F})$ be the set of formulae of $\mathcal{L}_\pi$ having the form

$$((\forall p)\xi p) \to \xi\phi$$

for all formulae ξ of $\mathcal{L}_\pi(*)$ and all formulae $\phi \in \mathcal{F}$. Similarly we can define $(\exists\mathcal{F})$ to be the set of formulae of $\mathcal{L}_\pi$ having the form

$$\xi\phi \to (\exists p)\xi p$$

for all ξ and ϕ as before. Also let $(C\mathcal{F})$ be the set of formulae having the form

$$(\exists p)(p \leftrightarrow \phi)$$

for all formulae $\phi \in \mathcal{F}$. Finally, when $\mathcal{F}$ is a set of formulae of $\mathcal{L}_\pi^-$ we let $(\forall^-\mathcal{F})$ be the set of all formulae of $\mathcal{L}^-$ having the form

$$((\forall p)\xi p) \to \xi\phi$$

for all formulae ξ of $\mathcal{L}_\pi^-(*)$ and all formulae $\phi \in \mathcal{F}$.

Proposition 16 *The standard theories ${}^mL_\pi + (\forall\mathcal{F})$, ${}^mL_\pi + (\exists\mathcal{F})$ and ${}^mL_\pi + (C\mathcal{F})$ are equivalent in the sense that they all have the same theorems.*

Proof. If $\phi \in \mathcal{F}$ then, as $\forall q(\xi q \to \exists p\xi p)$ is a theorem of ${}^mL_\pi$, $\xi\phi \to \exists p\xi p$ is a theorem of ${}^mL_\pi + (\forall\mathcal{F})$. Thus every formula in $(\exists\mathcal{F})$ is a theorem of ${}^mL_\pi + (\forall\mathcal{F})$.

If $\phi \in \mathcal{F}$ then $(\phi \leftrightarrow \phi) \to \exists p(p \leftrightarrow \phi)$ is a formula in $(\exists\mathcal{F})$. Thus every formula in $(C\mathcal{F})$ is a theorem of ${}^mL_\pi + (\exists\mathcal{F})$.

Finally, to complete the circle, it remains to prove that every formula in $(\forall\mathcal{F})$ can be proved in ${}^mL_\pi + (C\mathcal{F})$. This is a consequence of the following result. $\square$

Lemma 17 *For all formulae ξ of $\mathcal{L}_\pi(*)$ and ϕ of $\mathcal{L}_\pi$ the formula*

$$\exists p(p \leftrightarrow \phi) \to ((\forall p\xi p) \to \xi\phi)$$

is a theorem of ${}^mL_\pi$.

Proof. We work informally in ${}^mL_\pi$. First observe that by an easy induction on formulae ξ we get

$$(p \leftrightarrow \phi) \to (\xi p \to \xi\phi).$$

It follows that

$$(\forall p\xi p) \to \forall p((p \leftrightarrow \phi) \to \xi\phi).$$

As $\forall p((p \leftrightarrow \phi) \to \xi\phi) \to (\exists p(p \leftrightarrow \phi) \to \xi\phi)$ we get

$$(\forall p\xi p) \to (\exists p(p \leftrightarrow \phi) \to \xi\phi)$$

and hence the desired result $\exists p(p \leftrightarrow \phi) \to ((\forall p\xi p) \to \xi\phi)$. $\square$ When working in these theories it is natural to consider the elements of $\mathcal{F}$ as the 'terms' of sort π so that the formulae in $(\forall\mathcal{F})$ and $(\exists\mathcal{F})$ become natural quantifier axioms for π. Of course $(C\mathcal{F})$ is a kind of 'comprehension axiom scheme' for π.

For any language $\mathcal{L}$ let $\mathsf{form}(\mathcal{L})$ and $\mathsf{form}(\mathcal{L}^-)$ be the sets of all the formulae of $\mathcal{L}$ and $\mathcal{L}^-$ respectively. Let L^2 and L^{2-} be the standard theories $L_\pi + (\forall\mathsf{form}(\mathcal{L}_\pi))$ and $L^-_\pi + (\forall^-\mathsf{form}(\mathcal{L}^-_\pi))$ respectively. Similarly we can define ${}^mL^2$ and ${}^mL^{2-}$.

By structural recursion on ϕ we assign to each formula ϕ of $\mathcal{L}_\pi$ a formula ϕ^{w} of $\mathcal{L}^-_\pi$ using the following table.

ϕ	ϕ^{w}
θ	$(\forall p)((\theta \to p) \to p)$
$\perp$	$(\forall p)p$
$(\phi_1 \to \phi_2)$	$(\phi_1^{\mathsf{w}} \to \phi_2^{\mathsf{w}})$
$(\phi_1 \wedge \phi_2)$	$(\forall p)((\phi_1^{\mathsf{w}} \to \phi_2^{\mathsf{w}} \to p) \to p)$
$(\phi_1 \vee \phi_2)$	$(\forall p)((\phi_1^{\mathsf{w}} \to p) \to (\phi_2^{\mathsf{w}} \to p) \to p)$
$(\forall x : \sigma)\phi_0$	$(\forall x : \sigma)\phi_0^{\mathsf{w}}$
$(\exists x : \sigma)\phi_0$	$(\forall p)((\forall x : \sigma)(\phi_0^{\mathsf{w}} \to p) \to p)$

In this table σ can be any sort of $\mathcal{L}_\pi$ including π itself.

Proposition 18 *For any formula ϕ of $\mathcal{L}_\pi$, $\phi^{\mathsf{w}} \dashv\vdash_{L^2} \phi$, and for any set Γ of formulae of $\mathcal{L}_\pi$,*

$$\Gamma \vdash_{L^2} \phi \quad \textit{iff} \quad \Gamma^{\mathsf{w}} \vdash_{{}^mL^{2-}} \phi^{\mathsf{w}}.$$

7.1 The Russell-Prawitz modality

Let P be the defined unary connective of $\mathcal{L}_\pi^-$ given by the formula $\forall p((* \to p) \to p)$ of $\mathcal{L}_\pi^-(*)$.

Proposition 19 P *is a lax modality of* ${}^mL_\pi$ *and of* ${}^mL_\pi^-$.

Let $\mathsf{form}^{\mathsf{w}}(\mathcal{L}_\pi)$ be the set of **internal formulae of** $\mathcal{L}_\pi$; i.e. the smallest set $\mathcal{F}$ of formulae of $\mathcal{L}_\pi$ such that every variable p of sort π is in $\mathcal{F}$ and if ϕ is in $\mathcal{F}$ then so is $(\psi \to \phi)$ for every formula ψ of $\mathcal{L}_\pi$ and so is $(\forall x : \sigma)\phi$ for every variable x of any sort σ of $\mathcal{L}_\pi$, including π. Note that for any formula ϕ of $\mathcal{L}_\pi$ both ϕ^{w} and ϕ^{P} are internal formulae.

Let $L^{2\mathsf{w}}$ and ${}^mL^{2\mathsf{w}}$ be the standard theories $L_\pi + (\forall\mathsf{form}^{\mathsf{w}}(\mathcal{L}_\pi))$ and ${}^mL_\pi + (\forall\mathsf{form}^{\mathsf{w}}(\mathcal{L}_\pi))$ respectively.

Theorem 20 *For any formula* ϕ *of* $\mathcal{L}_\pi$, $\phi^{\mathsf{P}} \dashv\vdash_{{}^mL_\pi + (\bot \to \forall pp)} \phi^{\mathsf{w}}$, *and for any set* Γ *of formulae of* $\mathcal{L}_\pi$,

$$\Gamma \vdash_{L^2} \phi \quad \textit{iff} \quad \Gamma^{\mathsf{P}} \vdash_{{}^mL^{2\mathsf{w}}} \phi^{\mathsf{P}}.$$

Proposition 21 *Let* T *be a standard theory.*

1. $\mathsf{P} \leq_T \neg\neg$ *iff* $\vdash_T \neg\forall pp$, *if* $\vdash_T \bot \to \forall pp$,
2. $\neg\neg \leq_T \mathsf{P}$ *iff* $\vdash_T \forall p(\neg\neg p \to p)$,
3. $\neg\neg \equiv_T \mathsf{P}$ *iff* $\vdash_T (\neg\forall pp) \wedge \forall p(\neg\neg p \to p)$.

Proof.

1. By Proposition 3.3, $\mathsf{P} \leq_T \neg\neg$ iff $\vdash_T \neg\mathsf{P}\bot$. But if $\vdash_T \bot \to \forall pp$ then $\mathsf{P}\bot \leftrightarrow \forall pp$. The result follows.
2. First assume that $\neg\neg \leq_T \mathsf{P}$. Then $\neg\neg p \to \mathsf{P}p$ and so $\neg\neg p \to p$, as $\mathsf{P}p \to p$. Thus $\forall p(\neg\neg p \to p)$. Conversely, assume that (i) $\forall p(\neg\neg p \to p)$, (ii) $\neg\neg\phi$ and (iii) $\phi \to p$. Then, by (ii) and (iii) we get $\neg\neg p$ and so p by (i). Thus $\neg\neg\phi \to \forall p((\phi \to p) \to p)$.
3. Use 1 and 2 and the fact that $\forall p(\neg\neg p \to p)$ implies $(\bot \to \forall pp)$.

□

Proposition 22 *Let* J *be a lax modality of a standard theory* T.

1. *If* $\vdash_T \forall p(\mathsf{J}p \to p)$ *then* $\mathsf{J} \leq_T \mathsf{P}$.
2. *If* $\vdash_T (\forall p\xi p) \to \xi(\mathsf{J}\phi)$ *for every defined unary connective* ξ *and all formulae* ϕ *then* $\mathsf{P} \leq_T \mathsf{J}$.
3. *If* T *is a standard theory of* $\mathcal{L}$ *and for all formulae* ϕ *of* $\mathcal{L}$

$$\exists p(p \leftrightarrow \phi) \dashv\vdash_T (\mathsf{J}\phi \to \phi)$$

then $\mathsf{J} \equiv_T \mathsf{P}$.

Proof.

1. Let (i) $\forall p(\mathsf{J} \to p)$, (ii) $\mathsf{J}\phi$ and (iii) $\phi \to p$. Then, by (iii), $\mathsf{J}\phi \to \mathsf{J}p$, so that, by (ii) $\mathsf{J}p$ and so, by (i) p. Thus, from (i),(ii), we have shown $\forall p((\phi \to p) \to p)$; i.e. $\mathsf{P}\phi$.

2. If $\mathsf{P}\phi$; i.e. $\forall p((\phi \to p) \to p)$ then by the assumption $(\phi \to \mathsf{J}\phi) \to \mathsf{J}\phi$ so that, as $\phi \to \mathsf{J}\phi$, $\mathsf{J}\phi$. Thus $\mathsf{P}\phi \to \mathsf{J}\phi$.

3. Putting p for ϕ in the assumption we get that $\vdash_T (\mathsf{J}p \to p)$ so that by 1 $\mathsf{J} \leq_T \mathsf{P}$. As $\vdash_T \mathsf{J}\mathsf{J}\phi \to \mathsf{J}\phi$, by the assumption $\vdash_T \exists p(p \leftrightarrow \mathsf{J}\phi)$. Thus every theorem of ${}^m L_\pi + (C\mathcal{F})$ is a theorem of T, when $\mathcal{F}$ is the set of all formulae of $\mathcal{L}_\pi$ of the form $\mathsf{J}\phi$. So, by Proposition 16 every theorem of ${}^m L_\pi + (C\mathcal{F})$ is a theorem of T and hence by 2 $\mathsf{J} \leq_T \mathsf{P}$.

□

Theorem 23 *Let T be a standard theory for $\mathcal{L}_\pi$ that includes ${}^m L^{2\mathsf{w}}$. Then P is the unique, up to $\equiv_T$, lax modality J of T such that (i) $\vdash_T \forall p(\mathsf{J}p \to p)$ and (ii) for every formula ϕ of $\mathcal{L}_\pi$ there is an internal formula ψ of $\mathcal{L}_\pi$ such that $\mathsf{J}\phi \;\dashv\vdash_T\; \psi$.*

Proof. As $\vdash_T \forall p(\mathsf{P}p \to p)$ and $\mathsf{P}\phi$ is an internal formula of $\mathcal{L}_\pi$, P is a lax modality J satisfying (i) and (ii). Conversely suppose that J is a lax modality satisfying (i) and (ii). Then, by (i) and part 1 of the previous proposition, $\mathsf{J} \leq_T \mathsf{P}$. It remains to show that $\mathsf{P} \leq_T \mathsf{J}$ and this will be done by applying part 2 of the previous proposition. So let ξ be a defined unary connective of $\mathcal{L}_\pi$ and let ϕ be a formula of $\mathcal{L}_\pi$. Then by (ii) there is an internal formula ψ of $\mathcal{L}_\pi$ such that $\mathsf{J}\phi \;\dashv\vdash_T\; \psi$. As T includes ${}^m L^{2\mathsf{w}}$, $\vdash_T (\forall p \xi p) \to \xi\psi$. As $\vdash_T \mathsf{J}\phi \leftrightarrow \psi$, $\vdash_T \xi(\mathsf{J}\phi) \leftrightarrow \xi\psi$ so that $\vdash_T (\forall p\xi p) \to \xi(\mathsf{J}\phi)$. □ By combining this theorem with Proposition 21 we get the following result.

Theorem 24 *Let T be a standard theory for $\mathcal{L}_\pi$ that includes ${}^m L^{2\mathsf{w}} + \neg\forall pp + \forall p(\neg\neg p \to p)$. Then $\neg\neg$ is the unique, up to $\equiv_T$, lax modality J of T such that (i) and (ii) of the previous theorem hold.*

8 Conclusion

Recall the Russell-Prawitz definitions. These make sense in any standard theory T for $\mathcal{L}_\pi$ or $\mathcal{L}_\pi^-$ and define what we will call the **weak** logical operations:-

$$\begin{aligned}
[\phi_1 \wedge^{\mathsf{w}} \phi_2] &\equiv \forall p[(\phi_1 \to (\phi_2 \to p)) \to p] \\
[\phi_1 \vee^{\mathsf{w}} \phi_2] &\equiv \forall p[(\phi_1 \to p) \to ((\phi_2 \to p) \to p)] \\
[\neg^{\mathsf{w}}\phi] &\equiv \forall p[\phi \to p] \\
[\exists^{\mathsf{w}} x : \sigma\phi(x)] &\equiv \forall p[\forall x : \sigma(\phi(x) \to p) \to p]
\end{aligned}$$

There are several interesting cases to consider, depending on what is considered to be the intended interpretation of the sort π.

8.1 π is the impredicative sort of all propositions

We let T be the theory L^2. All the formulae of $\mathcal{L}_\pi$ represent propositions of sort π and the weak logical operations are equivalent to the standard ones.

8.2 π is the impredicative sort of all internal propositions

The **internal** propositions are those represented by internal formulae of $\mathcal{L}_\pi$ and we let T be the theory ${}^mL^{2\mathsf{w}}$. This is the case that is most relevant to the type theories of Lego and Coq, where the impredicative type *Prop* corresponds to the sort π. Here the weak logical operations are equivalent to the P versions of $\wedge$, $\vee$, $\neg$ and $\exists$ that are used in defining the P-translation.

8.3 π is the sort of all decidable propositions

The decidable propositions can be represented by the two formulae $\perp$ and $\perp \to \perp$ and we let T be the theory ${}^mL_\pi + (\forall\{\perp, (\perp \to \perp)\}) + \forall p(p \vee \neg p)$. The weak logical operations can here be characterised as those used in defining the translation $(\;)^N$:-

$$[\phi_1 \wedge^{\mathsf{w}} \phi_2] \quad \vdash\!\!\dashv_T \quad [(\phi_1 \to (\phi_2 \to \perp)) \to \perp]$$

$$[\phi_1 \vee^{\mathsf{w}} \phi_2] \quad \vdash\!\!\dashv_T \quad [(\phi_1 \to \perp) \to ((\phi_2 \to \perp) \to \perp)]$$

$$[\neg^{\mathsf{w}} \phi] \quad \vdash\!\!\dashv_T \quad [\phi \to \perp]$$

$$[\exists^{\mathsf{w}} x : \sigma\phi(x)] \quad \vdash\!\!\dashv_T \quad [\forall x : \sigma(\phi(x) \to \perp) \to \perp]$$

8.4 π is the sort of all J-stable propositions

We assume that T is a standard theory for $\mathcal{L}_\pi$ and that J is a lax modality of T that has the property that for all formulae ϕ of $\mathcal{L}_\pi$

$$\exists p(p \leftrightarrow \phi) \;\; \vdash\!\!\dashv_T \;\; (\mathsf{J}\phi \to \phi).$$

The J-stable propositions are those propositions represented by formulae of the form $\mathsf{J}\phi$. Here the weak logical operations are simply the J-versions of $\wedge$, $\vee$, $\neg$ and $\exists$.

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