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Donald J. MacInnes
Barnaby P. Hilken
David E. Rydeheard

Department of Computer Science
University of Manchester

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Donald J. MacInnes
Barnaby P. Hilken
David E. Rydeheard

Department of Computer Science
University of Manchester
Oxford Road, Manchester, UK.
`dmacinnes@cs.man.ac.uk`

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Abstract

In a recent paper Hilken presents a generalisation of the adjunction and duality between frames and topological spaces to give a similar result which applies to a certain class of intuitionistic monomodal logics. This paper describes a further generalisation of this result by extending the adjunction and duality to polymodal languages. A *labelled-relational space* is defined to be a topological space carrying a collection of binary relations on the points and a *polymodal frame* is a frame with a corresponding collection of pairs of operations, $[i]$ and $\langle i \rangle$, on the elements which satisfy various axioms. With the appropriate definitions of morphisms both of these classes become categories and a contravariant adjunction is constructed between them. This adjunction then restricts to a duality between subcategories in the image of the adjunction.

Once this indexed generalisation has been completed the report proceeds to consider how the labelled components of the spaces and of the frames can be re-indexed. Two covariant adjunctions are constructed which permit this relabelling.

Composition of the various functors and adjunctions which have been constructed is considered and the Beck-Chevalley conditions are mentioned.

Chapter 1

Introduction

1.1 Background

The study of classical modal logic begins with the notions of necessity and possibility, and various other related ideas; for example, provability and consistency within some formal system of arithmetic or moral obligation and acceptability in the case of deontic logic. The modal language is defined as an extension of the Boolean propositional language and is obtained by adding a batch of new unary connectives called *modalities*, one for each element i of some index set I. The elements of I are called *labels* and I itself is called the *signature* of the language. These new connectives are used to represent the new notions (necessity and possibility, or whatever else) and are usually referred to as the *box* and *diamond* connectives because they are written as $[i]$ and $\langle i \rangle$. There are various reasons for not limiting the modal language to having just one box and one diamond, an important example being tense logic in which one pair of modalities deals with the future and another pair with the past. Indeed sometimes tense logics have two other pairs of modalities which are used to describe the immediate future and immediate past.

Before the work of Kripke in the 1960s most research in modal logic consisted of philosophical arguments as to what axioms these modalities ought to satisfy and the language was mainly monomodal, which is to say that the signature was just a singleton set. Kripke's universal semantics, which applies to all normal modal languages, transfers these considerations to simple properties of binary relations by introducing a particular kind of relational structure usually referred to as a *transition system* or a *Kripke frame*. These consist of a set of 'worlds' equipped with a collection of binary relations, $\xrightarrow{i}$, one for each label of the signature of the language. The powerset of each relational structure has a natural algebraic structure — the *modal algebra* — onto which Kripke's semantics can be transferred without loss of any information. On the modal algebra the semantics for the modalities is given by

$$\begin{aligned} [i](\phi) &= \{x \mid \forall y\,.\, x \xrightarrow{i} y \Rightarrow y \in \phi\} \\ \langle i \rangle(\phi) &= \{x \mid \exists y\,.\, x \xrightarrow{i} y \ \ \& \ \ y \in \phi\}. \end{aligned} \tag{1.1}$$

Given an appropriate proof theory, it is well know that for many modal systems, (e.g. K, D, T, B, S4, S5 and so on) a completeness theorem can be proved. The soundness theorem makes good use of the modal algebra just mentioned and the adequacy result demonstrates that the maximal consistent sets of each modal algebra have a natural relational structure. In fact, with the correct definitions of morphism both of these classes become categories, between which there is a *duality*, (or *contravariant equivalence*). It is this important result which lies behind the subject of correspondence theory — a modal algebra satisfies some axiom if and only if the relational structure has some property. In turn, it is the fact that there is such a rich correspondence theory that makes Kripke's semantics so useful and intuitively appealing. A full account of this brief summary can be found in [7].

Intuitionistic modal logic is currently in a similar state to that of its classical counterpart before Kripke. There is no widely accepted semantic framework and neither is there general agreement as to which axioms ought to be satisfied. Work has been done which attempts to fuse Kripke's semantics for modal logic with his similar semantics for intuitionistic logic, but in [2] Hilken opts for a different approach. There he concentrates on the duality aspect mentioned above and sets about merging two dualities: that between monomodal algebras and relational structures (carrying a single relation) and that between *frames* and *topological spaces*. A detailed account of this second duality, which lies behind many important results in lattice theory and topos theory, can be found in [3].

Hilken combines the semantic models of the modal logic by considering a set carrying both a relation and a topology and, in line with the topological semantics of intuitionistic logic, he interprets propositions as open sets in the new model. As the sets in (1.1) are in general not open, the interior of each is taken as the new definition of $\Box$ and $\Diamond$. This amounts to defining a pair of unary functions on the frame of open sets of the space. These new models can be pictured as putting a topology on the traditional set of worlds and the usual (Kripke) models can be recovered by using the discrete topology. The main result of [2] is the construction of a contravariant adjunction between the categories of *relational spaces* and *modal frames* which demonstrates that modal frames are the algebraic version of relational spaces in the same sense that unadorned frames are the algebraic counterpart of standard topological spaces. This adjunction then restricts to a duality between certain subcategories.

The first big result of this paper is to extend this adjunction to the polymodal case and show that it again restricts to a duality between appropriate subcategories. Having provided an indexed version of the adjunction it is natural to consider the question of relabelling both the relational spaces and the modal frames. In view of this, two more adjunctions are presented which allow for this relabelling. We end by considering how these adjunctions fit together and briefly consider a few applications to computer science.

1.2 General Re-indexing

The move from considering single objects in a category to considering indexed families of objects is commonplace in category theory and there are various ways of doing this. In the next few paragraphs we shall briefly mention two of them.

Suppose we are given some base category B (which is often Set), then certain functors $F : \mathsf{A} \longrightarrow \mathsf{B}$ can be used to provide a way of considering the objects $A \in \mathsf{A}$ as indexed families $(A_i)_{i \in I}$, which are indexed by the object $I = FA$ from B. This approach to indexing leads to the theory of *fibrations*. The usefulness of this theory is that when a functor is a fibration, it provides the necessary conditions in which arrows in the base category B (e.g. mappings between sets) can always be lifted to produce re-indexing arrows in A. More on fibrations can be found in [8].

Another way of dealing with indexed families is to move from a category A to the *slice category* A/X whose objects are arrows $f : A \longrightarrow X$ from A, all with codomain X, and whose arrows are the commuting triangles $h : f \longrightarrow g$ as follows:

$$\begin{tikzcd} A \arrow[dr, "f"] \arrow[dd, "h"'] & \\ & X \\ B \arrow[ur, "g"'] & \end{tikzcd}$$

This technique provides a way of keeping all the details of the indexing internal to the category A, because each object A can be thought of as the disjoint union of the $f^{\leftarrow}(x)$ for $x \in X$. Re-indexing in this context amounts to a functor between slices. Given an arrow $f : X \longrightarrow Y$ in A the functor $\Sigma_f : \mathsf{A}/X \longrightarrow \mathsf{A}/Y$ takes each A/X object $g : A \longrightarrow X$ to $f \circ g$ and each A/X arrow $h : g \longrightarrow g'$ to $h : f \circ g \longrightarrow f \circ g'$. All of this can be found in [6], together with further details about the theory of slice categories.

Although these methods of dealing with re-indexing are useful in some contexts, they are not used in this report. The main reason for this is that the indexed families which we shall encounter are actually essential facilities of the objects of each category and not just indexed families of objects. This is an important difference and so our indexed families are treated more informally. This will become clear once the objects of our study have been defined in the next chapter.

1.3 Terminology

To finish this chapter it is worth mentioning a few details about the terminology and notation which will be assumed in the course of this work. For our purposes a *frame* is a structure

$$(A, \leqslant, \wedge, \bigvee, \top, \bot)$$

where $(A, \leqslant)$ is a partially ordered set (poset) which is closed under finite meets and arbitrary joins. This means that in each frame there is unique top element and a unique bottom element. These will, as usual, be denoted by $\top$ and $\bot$ respectively. Each frame also satisfies the *frame distributive law* — for $a \in A$ and $S \subseteq A$ we have

$$a \wedge \bigvee S = \bigvee \{a \wedge s | \, s \in S\}.$$

As we shall later equip frames with more structure, from now on frames will be denoted as just $(A, \leqslant)$. A frame morphism is a monotone map which preserves finite meets and all joins. Note that, as they are closed under arbitrary joins, frames are also closed under arbitrary meets (see [1]). The distinction is that frame morphisms are not required to preserve arbitrary meets, only finite ones. However, we shall make good use of the fact that frames have the facility to form arbitrary meets in Chapter 3. In the literature frames are also known as *locales* and *complete Heyting algebras*. In terms of objects these terms all mean the same thing, but there are differences between the corresponding categories, so these other names will not crop up again.

Finally, we shall not use the word 'frame' in the context of 'Kripke frame'; indeed this is the last time that Kripke frames will be mentioned.

1.4 Notation

Following common practice in category theory we write 1_X for the identity arrow of the object X and also write 1 for the identity functor. **2** is the two-element frame $\{\bot \leqslant \top\}$. For a topological space X and $S \subseteq X$ we write S° for the topological interior and $\overline{S}$ for the closure of S. If $f : X \longrightarrow Y$, then $f^\leftarrow$ is the inverse-image map of f. $\mathcal{P}$ is the powerset operator. Parentheses are regarded as punctuation and so will be used, or omitted, for reasons of clarity alone. Occasionally, for the sake of brevity, we shall use just the name of the carrying set of the structure rather than listing all the details.

Chapter 2

The Extension to Polymodal Languages

2.1 Labelled-Relational Spaces

The first step in the move to a polymodal language is to define the basic concrete structures that will constitute the semantic side of the first adjunction. These are no more than topological spaces furnished with a collection of binary relations.

Definition. A **labelled-relational space** $(X, \tau, \xrightarrow{i})$ is a topological space (X, τ) together with a collection of binary relations, $\xrightarrow{i}$, on the points with $i \in I$ for some index set I.

Definition. A **continuous labelled-relational function**

$$f : (X, \tau, \xrightarrow{i}) \longrightarrow (Y, \sigma, \xrightarrow{i})$$

is a continuous function $f : (X, \tau) \longrightarrow (Y, \sigma)$ such that for each label $i \in I$ we have

$$\forall x \forall y (x \xrightarrow{i} y \Rightarrow fx \xrightarrow{i} fy). \tag{2.1}$$

A **continuous p-morphism** is a continuous labelled-relational function such that for each $i \in I$ we have

$$\forall x \forall y ((fx \xrightarrow{i} y \;\;\&\;\; y \in U \in \sigma) \Rightarrow \exists x' (x \xrightarrow{i} x' \;\;\&\;\; fx' \in U)). \tag{2.2}$$

Usually a p-morphism is a function which preserves relations and satisfies

$$\forall x \forall y ((fx \xrightarrow{i} y) \Rightarrow \exists x' (x \xrightarrow{i} x' \;\;\&\;\; fx' = y)), \tag{2.3}$$

but this is not what we have here. Instead, a continuous p-morphism satisfies a slightly weaker condition which gets arbitrarily close to satisfying (2.3) by taking $U \ni y$ to be arbitrarily small. Of course, if the topology is discrete then we regain the usual p-morphism condition (2.3).

Labelled-relational spaces and continuous p-morphisms form a category, which will be called LRelSp.

2.2 Polymodal Frames

In the spirit of the duality between frames and topological spaces, we wish to study the algebraic nature of the open sets of a space. The collection of open sets forms a frame, but we also need some extra structure with which the relations can be studied. This structure is a family of pairs of functions on the frame, whose definition reflects the modal viewpoint.

Definition. A **polymodal frame** $(A, \leqslant, [i], \langle i \rangle)$ is a frame $(A, \leqslant)$ together with pairs of monotone maps $[i], \langle i \rangle : A \longrightarrow A$ which, for each label i of some index set I, satisfy

$$\begin{aligned} \top &\leqslant [i]\ \top \\ [i]\ a \wedge [i]\ b &\leqslant [i]\ (a \wedge b) \\ \langle i \rangle\ a \wedge [i]\ b &\leqslant \langle i \rangle\ (a \wedge b) \\ \langle i \rangle \perp &\leqslant \perp. \end{aligned} \tag{2.4}$$

Definition. A **polymodal frame morphism**

$$f : (A, \leqslant, [i]^A, \langle i \rangle^A) \longrightarrow (B, \preccurlyeq, [i]^B, \langle i \rangle^B)$$

is a frame morphism $f : (A, \leqslant) \to (B, \preccurlyeq)$ which, for each label $i \in I$, satisfies

$$f([i]^A\, a) \preccurlyeq [i]^B f(a) \tag{2.5}$$

$$f(\langle i \rangle^A a) \preccurlyeq \langle i \rangle^B f(a). \tag{2.6}$$

The objects and arrows of the second category which will be studied are the polymodal frames and polymodal frame morphisms. This category will be called PMFrm. It is the relationship between these two categories, LRelSp and PMFrm, that is studied in this chapter. We shall begin by proving two lemmas whose purpose is to define the functor to which we shall later construct a right adjoint.

Lemma 1. *Let $(X, \tau, \xrightarrow{i})$ be a labelled-relational space. If the box and diamond operations $[i], \langle i \rangle : \tau \longrightarrow \tau$ are defined to be*

$$[i]\ (U) = \{x \in X | \forall y \in X (x \xrightarrow{i} y \ \Rightarrow \ y \in U)\}^\circ \tag{2.7}$$

$$\langle i \rangle\ (U) = \{x \in X | \exists y \in X (x \xrightarrow{i} y \ \ \& \ \ y \in U)\}^\circ \tag{2.8}$$

then $(\tau, \subseteq, [i], \langle i \rangle)$ is a polymodal frame.

Proof. Each of the four conditions of (2.4) follow by routine calculation.

$$[i]\,X = \{x|\,\forall y \in X(x \xrightarrow{i} y \;\Rightarrow\; y \in X)\}^\circ = X.$$

$$\begin{aligned}[i]\,U \cap [i]\,V &\subseteq \{x \in X|\forall y \in X(x \xrightarrow{i} y \;\Rightarrow\; y \in (U \cap V))\}^\circ \\ &= [i]\,(U \cap V).\end{aligned}$$

$$\begin{aligned}\langle i\rangle\,U \cap [i]\,V &\subseteq \{x \in X|\exists y \in X(x \xrightarrow{i} y \;\&\; y \in (U \cap V))\}^\circ \\ &= \langle i\rangle\,(U \cap V).\end{aligned}$$

$$\langle i\rangle\,\emptyset = \{x \in X|\exists y \in X(x \xrightarrow{i} y \;\&\; y \in \emptyset)\}^\circ = \emptyset.$$

□

Lemma 2. *If $f : (X, \tau, \xrightarrow{i}) \longrightarrow (Y, \sigma, \xrightarrow{i})$ is a continuous p-morphism, then the inverse-image map $f^{\leftarrow} : (\sigma, \subseteq, [i]^Y, \langle i\rangle^Y) \longrightarrow (\tau, \subseteq, [i]^X, \langle i\rangle^X)$ is a polymodal frame morphism.*

Proof. It is routine to see that $f^{\leftarrow}$ is a frame morphism, so we need only show that for each $i \in I$ we have $f^{\leftarrow}([i]^Y U) \subseteq [i]^X f^{\leftarrow}(U)$ and $f^{\leftarrow}(\langle i\rangle^Y U) \subseteq \langle i\rangle^X f^{\leftarrow}(U)$.

Fix i and let $x \in f^{\leftarrow}([i]^Y U)$ so that $fx \in [i]^Y U$. Now, if $x \xrightarrow{i} x'$, then $fx \xrightarrow{i} fx'$ and so $fx' \in U$, i.e. $x' \in f^{\leftarrow}(U)$. Thus we have that

$$f^{\leftarrow}([i]^Y U) \subseteq \{x|\forall x'(x \xrightarrow{i} x' \Rightarrow x' \in f^{\leftarrow}(U))\}.$$

However $f^{\leftarrow}([i]^Y U)$ is open, so is a subset of $[i]^X f^{\leftarrow}(U)$.

Fix i again and let $x \in f^{\leftarrow}(\langle i\rangle^Y U)$ so that $fx \in \langle i\rangle^Y U$. This means that there is some y for which $fx \xrightarrow{i} y$ and so there is some x' such that $x \xrightarrow{i} x'$, thus $fx \xrightarrow{i} fx'$ which means $fx' \in U$, i.e. $x' \in f^{\leftarrow}(U)$. Thus we have that

$$f^{\leftarrow}(\langle i\rangle^Y U) \subseteq \{x|\exists x'(x \xrightarrow{i} x' \;\&\; x' \in f^{\leftarrow}(U))\}.$$

However $f^{\leftarrow}(\langle i\rangle^Y U)$ is open, so is a subset of $\langle i\rangle^X f^{\leftarrow}(U)$. □

The results of last two lemmas are worth recording clearly as follows.

Proposition 3. *The map $\mathcal{O} : \mathsf{LRelSp} \longrightarrow \mathsf{PMFrm}$ given by*

$$\mathcal{O}(X, \tau, \xrightarrow{i}) = (\tau, \subseteq, [i], \langle i\rangle) \tag{2.9}$$

$$\mathcal{O}f = f^{\leftarrow} \tag{2.10}$$

is a contravariant functor.

2.3 Polymodal Frame Points

In the adjunction between frames and topological spaces, the spaces are created from the frames by forming *frame points*. These have various characterisations, all of which are equivalent. A frame point of $(A, \leqslant)$ can be represented as any of the following:

- a completely prime filter
- a principal ideal
- a frame morphism $p : A \longrightarrow \mathbf{2}$.

The details of the relationships between these gadgets can be found in [3].

For our purposes the most useful of these characterisations is the last one. However, as polymodal frames have more structure than plain frames, merely forming a space of frame points does not give a space with the right properties. As such, taking inspiration from the monomodal case, we define a set of *polymodal frame points*. These consist of a frame point and indexed family of elements of the frame, one for each $i \in I$. As in the monomodal case, the frame point describes which open sets the point inhabits and each element of the indexed family can be thought of as being the interior of the set of unrelated points for the relation corresponding the that label.

Definition. Let $(A, \leqslant, [i], \langle i \rangle)$ be a polymodal frame. A **pre-point** of this frame is a pair $(p, \mathcal{A})$ where $p : A \longrightarrow \mathbf{2}$ is a frame point and $\mathcal{A} = \{a_i \mid i \in I\}$ is an indexed family of elements of A, one for each label i, which satisfies

$$p(\langle i \rangle\, a_i) = 0 \tag{2.11}$$

for each $i \in I$.

Now let us define the i-**projection** of each pre-point $(p, \mathcal{A})$ to be the pair (p, a_i). (Here a_i is clearly the i^{th} element of the indexed family $\mathcal{A}$.) We shall say that two i-projections are **related**, $(p, a_i)\, S_i\, (q, b_i)$ if and only if

$$q(a_i) = 0 \tag{2.12}$$

$$\forall c \in A.\, p([i]\ c) \leqslant q(c). \tag{2.13}$$

The S_i relations allow us to define a batch of induced binary relations, R_i, on the pre-points themselves by saying that two pre-points are i-**related**, $(p, \mathcal{A})\, R_i\, (q, \mathcal{B})$, if and only if their i-projections are S_i related.

Note that although the i-projections are called projections, we shall not consider any projection maps. The name is used only to give an instant picture of which of the indexed family of elements is important for each relation.

Definition. The set of **polymodal frame points** $\mathbb{P}^A$ is the *largest* set P of pre-points which satisfies $(p, \mathcal{A}) \in P$ if and only if

$$(p, \mathcal{A}) \in P \ \ \&\ \ c \nleqslant a_i \Longrightarrow \exists (q, \mathcal{B}) \in P.\, (p, \mathcal{A})\, R_i^A\, (q, \mathcal{B}) \ \ \&\ \ q(c) = 1. \tag{2.14}$$

Condition (2.14) captures the property that each a_i is the interior of the set of i-unrelated points. The relations on the polymodal frame points are just the R_i^A, so we now only need give the topology.

Definition. The **unit** $\phi_A : A \longrightarrow \mathcal{P}(\mathbb{P}^A)$ is defined as

$$\phi_A(b) = \{(p, \mathcal{A}) \in \mathbb{P}^A |\, p(b) = 1\}. \tag{2.15}$$

The **topology** τ^A on $\mathbb{P}^A$ is the image of ϕ_A.

Lemma 4. *The image of ϕ_A is a topology.*

Proof. This is a standard result, which, by the definition of ϕ_A, depends only on the frame point of each polymodal frame point. There is a proof in [3] that the image of ϕ_A is closed under arbitrary unions, so below is the completely analogous proof for binary intersections.

$$\begin{aligned} p \in \phi_A(b) \cap \phi_A(c) &\iff p(b) = 1 \text{ and } p(c) = 1 \\ &\iff p(b \wedge c) = 1 \\ &\iff p \in \phi_A(b \wedge c). \end{aligned}$$

□

Lemma 5 (Unit on the frame side). *The unit*

$$\phi_A : (A, \leqslant, [\![i]\!], \langle\!\langle i\rangle\!\rangle) \longrightarrow (\tau^A, \subseteq, [\![i]\!]^{R^A}, \langle\!\langle i\rangle\!\rangle^{R^A}) \tag{2.16}$$

is a polymodal frame morphism.

Proof. That ϕ_A is a frame morphism is standard, so we need only check (2.5) and (2.6).

Fix some label i and let $(p, \mathcal{A}) \in \phi_A([\![i]\!]\, c)$, so that $p([\![i]\!]\, c) = 1$. If $(p, \mathcal{A})\, R_i^A\, (q, \mathcal{B})$ then $q(c) \geqslant p([\![i]\!]\, c) = 1$ and so $(q, \mathcal{B}) \in \phi_A(c)$. Therefore we have that

$$\phi_A([\![i]\!]\, c) \subseteq \{x |\, \forall y(x\, R_i^A\, y \Rightarrow y \in \phi_A(c))\},$$

but $\phi_A([\![i]\!]\, c)$ is open, thus $\phi_A([\![i]\!]\, c) \subseteq [\![i]\!]^{R^A} \phi_A(c)$.

Fix i again and let $(p, \mathcal{A}) \in \phi_A(\langle\!\langle i\rangle\!\rangle c)$, so that $p(\langle\!\langle i\rangle\!\rangle c) = 1$. Then $c \nleqslant a_i$, so by (2.14) there is some $(q, \mathcal{B})$ such that $(p, \mathcal{A})\, R_i^A\, (q, \mathcal{B})$ and $q(c) = 1$, which means $(q, \mathcal{B}) \in \phi_A(c)$. Therefore

$$\phi_A(\langle\!\langle i\rangle\!\rangle c) \subseteq \{x |\, \exists y(x\, R_i^A\, y \;\;\&\;\; y \in \phi_A(c))\},$$

but $\phi_A(\langle\!\langle i\rangle\!\rangle c)$ is open and so $\phi_A(\langle\!\langle i\rangle\!\rangle c) \subseteq \langle\!\langle i\rangle\!\rangle^{R^A} c$. □

2.4 The Adjunction

This section is devoted to the proof of the contravariant adjunction between the categories that have been introduced. Most of the results are achieved by making the necessary adjustments to the proofs presented in [2] so that they now cope with the indexed components of the objects in the two categories. We begin by defining the object assignment of the proposed right adjoint.

Theorem 1. *The map* $(A, \leqslant, [i], \langle i \rangle) \longmapsto (\mathbb{P}^A, \tau^A, R_i^A)$ *is the object part of a contravariant functor, adjoint to* $\mathcal{O}$ *on the right, with unit* ϕ_A *on the frame side.*

We begin the proof of this result by defining the arrows which arise from the universal property.

Definition. Let $(A, \leqslant, [i]^A, \langle i \rangle^A)$ be a polymodal frame, $(X, \tau, \xrightarrow{i})$ a labelled-relational space and $f : (A, \leqslant, [i]^A, \langle i \rangle^A) \longrightarrow (\tau, \subseteq, [i]^{\rightarrow}, \langle i \rangle^{\rightarrow})$ a polymodal frame morphism. Define $f^\sharp : X \longrightarrow \mathbb{P}^A$ by

$$f^\sharp(x) = (p^x, \mathcal{A}^x) \quad \text{where} \quad \begin{aligned} p^x(b) &= 1 \text{ iff } x \in f(b) \\ a_i^x &= \bigvee \{c \in A \mid \forall y \in f(c) \,.\, x \stackrel{i}{\not\rightarrow} y\} \end{aligned} \tag{2.17}$$

The next three lemmas will show that $(p^x, \mathcal{A}^x)$ is a polymodal frame point.

Lemma 6. *For all* $x \in X$, $f^\sharp(x)$ *is a pre-point.*

Proof. Fix i. If $p^x(\langle i \rangle^A a_i^x) = 1$, then $x \in f(\langle i \rangle^A a_i^x) \subseteq \langle i \rangle^{\rightarrow} f(a_i^x)$ so there is some $y \in X$ such that $x \xrightarrow{i} y$ and $y \in f(a_i^x)$. However

$$f(a_i^x) = \bigcup \{f(c) \mid \forall y \in f(c) \,.\, x \stackrel{i}{\not\rightarrow} y\},$$

which contradicts $x \xrightarrow{i} y$ and so we conclude that $p^x(\langle i \rangle^A a_i^x) = 0$. □

Lemma 7. *For each label* i*, if* $x, y \in X$ *are such that* $x \xrightarrow{i} y$ *then* $f^\sharp(x) \, R_i^A \, f^\sharp(y)$*.*

Proof. Fix i. If $p^y(a_i^x) = 1$ then $y \in f(a_i^x) = \bigcup \{f(c) \mid \forall y' \in f(c) \,.\, x \stackrel{i}{\not\rightarrow} y'\}$ contradicting $x \xrightarrow{i} y$ and so $p^y(a_i^x) = 0$.

If $p^x([i]^A b) = 1$ then $x \in f\, [i]^A b \subseteq [i]^{\rightarrow} f(b)$, thus $y \in f(b)$ i.e. $p^y(b) = 1$ and so $p^x([i]^A b) \leqslant p^y(b)$. □

Lemma 8. *If* $x \in X$ *then* $f^\sharp(x) \in \mathbb{P}^A$*.*

Proof. Let $P = \{f^\sharp(x') \mid x' \in X\}$. We shall show that P satisfies (2.14) and so $f(x) \in P \subseteq \mathbb{P}^A$. If $(p^x, \mathcal{A}^x) \in P$ and $b \not\leqslant a_i^x = \bigvee \{c \mid \forall y \in f(c) \,.\, x \stackrel{i}{\not\rightarrow} y\}$ then there's some $y \in f(b)$ such that $x \xrightarrow{i} y$. But then if we let $(p^y, a_i^y) = f^\sharp(y)$, we have $p^y(b) = 1$ and $(p^x, a_i^x) \, R_i^A \, (p^y, a_i^y)$ by Lemma 7. □

Lemma 7 shows that $f^\sharp$ preserves the relations; to show that it is a continuous p-morphism we need to show continuity and property (2.2).

Lemma 9. *For all $a \in A$, $(f^\sharp)^{\leftarrow}(\phi_A(a)) = f(a)$ and so $f^\sharp$ is continuous.*

Proof. $x \in (f^\sharp)^{\leftarrow}(\phi_A(a))$ iff $f^\sharp(x) \in \phi_A(a)$ iff $p^x(a) = 1$ iff $x \in f(a)$ and so the result follows. □

Lemma 10. *If $x \in X$ and $(p, \mathcal{A}) \in \phi_A(b) \in \tau^A$ is such that $f^\sharp(x)\ R_i^A\ (p, \mathcal{A})$ then there exists $y \in X$ such that $x \xrightarrow{i} y$ and $f^\sharp(y) \in \phi_A(b)$.*

Proof. As $p(a_i^x) = 0$ and $p(b) = 1$ we have $b \not\leqslant a_i^x = \bigvee\{c|\, \forall y \in f(c)\,.\, x \stackrel{i}{\not\to} y\}$, and so there is some $y \in f(b)$ with $x \xrightarrow{i} y$. Now, by Lemma 9, $f^\sharp(y) \in \phi_A(b)$. □

To finish the proof of the adjunction we now only have to show that $f^\sharp$ is the unique arrow with the required property.

Lemma 11. *If $g : (X, \tau, \xrightarrow{i}) \longrightarrow (\mathbb{P}^A, \tau^A, R_i^A)$ satisfies $g^{\leftarrow}\phi_A = f$, then $g = f^\sharp$.*

Proof. For all $x \in X$ let $(q^x, \mathcal{B}^x) = g(x)$; we shall prove that $q^x = p^x$ and $\mathcal{B}^x = \mathcal{A}^x$.

$p^x(c) = 1$ iff $x \in f(c)$ iff $x \in g^{\leftarrow}\phi_A(c)$ iff $g(x) \in \phi_A(c)$ iff $q^x(c) = 1$ and so we have shown $p^x = q^x$.

If $a_i^x \not\leqslant b_i^x$ then (from the definition of a_i^x) there is some $c \not\leqslant b_i^x$ such that for all $y \in f(c)$ we have $x \stackrel{i}{\not\to} y$. Since $(q^x, \mathcal{B}^x) \in \mathbb{P}^A$ there is $(q, \mathcal{B}) \in \mathbb{P}^A$ such that $(q^x, \mathcal{B}^x)\, R_i^A\, (q, \mathcal{B})$ and $q(c) = 1$, i.e. $(q, \mathcal{B}) \in \phi_A(c)$. Now, as g satisfies condition (2.2), there is some y such that $x \xrightarrow{i} y$ and $g(y) \in \phi_A(c)$ which is to say $y \in f(c)$. Contradiction, so $a_i^x \leqslant b_i^x$.

If $b_i^x \not\leqslant a_i^x$ then there is some $y \in f(b_i^x)$ such that $x \xrightarrow{i} y$. Then $g(x)\, R_i^A\, g(y)$ and so $q^y(b_i^x) = 0$ i.e. $y \notin f(b_i^x)$ and this is again a contradiction, so $b_i^x \leqslant a_i^x$. □

This completes the proof of Theorem 1. To finish this section we present explicitly the arrow assignment of the contravariant functor from polymodal frames to labelled-relational spaces and give the topological unit.

Corollary 12. *The functor $\mathcal{T}$: $\mathsf{PMFrm} \longrightarrow \mathsf{LRelSp}$, right adjoint to $\mathcal{O}$, is defined on arrows $f : A \longrightarrow B$ by*

$$\mathcal{T}(f)(p, \{a_i|\, i \in I\}) = (p \circ f, \{\bigvee\{b|\, f(b) \leqslant a_i\}\,|\, i \in I\}).$$

The topological unit ψ of the adjunction is defined by

$$\begin{aligned} \psi_X(x) = (p^x, \{a_i^x\,|\, i \in I\}) \quad \text{where} \quad p^x(U) = 1 \ \text{ iff } \ x \in U \\ a_i^x = \{y|\, x \not R_i^A y\}^\circ. \end{aligned} \tag{2.18}$$

2.5 Duality Theorem

If we consider a labelled-relational space, apply $\mathcal{O}$ to take its polymodal frame of open sets, then apply $\mathcal{T}$ to convert this frame back into a space we might not end up with a space which is isomorphic to the one with which we started. Similarly, if we start with a polymodal frame, construct its space of polymodal frame points, then take the frame of open sets, the resulting frame may not be isomorphic to the original one. It is therefore useful to know under what conditions we do have isomorphisms; i.e. when are the units are iso? The next two propositions will show that ϕ_A is iso when A is (isomorphic to) the frame of opens of a labelled-relational space and ψ_X is iso when X is (isomorphic to) the space of polymodal frame points of a frame. It is useful to have names for these situations, so we begin with a definition.

Definition. A polymodal frame A is **modally spatial** if ϕ_A is an isomorphism. A labelled-relational space X is **modally localic** if ψ_X is an isomorphism.

Proposition 13. *If X is a labelled-relational space, then $\mathcal{O}(X)$ is modally spatial.*

Proof. As the topology on $\mathcal{T}(A)$ is the image of ϕ_A, ϕ_A is therefore surjective and so is epic in LRelSp. By the triangle identities $\phi_{\mathcal{O}(X)}$ is split monic. Hence, ϕ_A is iso. □

Proposition 14. *If A is a polymodal frame, then $\mathcal{T}(A)$ is modally localic.*

Proof. By the triangle identities we have that $\mathcal{T}(\phi_A) \circ \psi_{\mathcal{T}(A)} = 1_{\mathcal{T}(A)}$; it remains to show that $\psi_{\mathcal{T}(A)} \circ \mathcal{T}(\phi_A) = 1_{\mathcal{TOT}(A)}$.

Let $(p, \mathcal{U}) \in \mathcal{TOT}(A)$ and let

$$x = \mathcal{F}(\phi_A)(p, \mathcal{U}) = (p \circ \phi_A, \{\bigvee\{a|\, \phi_A(a) \subseteq U_i\}|\, i \in I\})$$

so that $\psi_X(x) = (p^x, \{a_i^x \,|\, i \in I\})$ as in (2.18).

$p^x \circ \phi_A(a) = 1$ iff $x \in \phi_A(a)$ iff $p \circ \phi_A(a) = 1$, so $p^x = p$.

$\phi_A(_i) \subseteq U_i$ iff $a \leqslant \bigvee\{a|\, \phi_A(a) \subseteq U_i\}$, and so

$$\begin{aligned} \phi_A(a) \subseteq a_i^x &\iff \phi_A(a) \subseteq \{y|\, x \not R_i^A\, y\} \\ &\iff \forall (q, \mathcal{B}) \in \mathbb{P}^A (x\, R_i^A\, (q, \mathcal{B}) \Rightarrow q(a) = 0). \end{aligned}$$

If $a \leqslant \bigvee\{a|\, \phi_A(a) \subset U_i\}$ and $x\, R_i^A\, (q, \mathcal{B})$ then $q(\bigvee\{a|\, \phi_A(a) \subset U_i\}) = 0$ so $q(a) = 0$. Therefore $U_i \subseteq a_i^x$.

If $a \not\leqslant \bigvee\{a|\, \phi_A(a) \subset U_i\}$ then, by (2.14), there is some $(q, \mathcal{B}) \in \mathbb{P}^A$ such that $x\, R_i^A\, (q, \mathcal{B})$ and $q(a) = 1$. Therefore $a_i^x \subseteq U_i$. □

Before reaching the end of this chapter, a few aspects of this result should be mentioned. When the set of labels is empty, the objects of our study become merely plain topological spaces and frames. It is therefore not surprising that the adjunction and duality reduce to the well known equivalence result between the categories of topological spaces and frames that was mentioned in the introduction. Also, when the index set I is just a singleton we recover the adjunction and duality of [2], so the work done here is a genuine extension of both of these previous results.

In section 3.1.1 there is a description of the terminal object in the category LRelSp of labelled-relational spaces. It turns out that the size of the set of labels plays an integral part in determining what this object is. As will become apparent, the terminal object in this category is non-trivial and this fact has some interesting consequences. In particular, considerations of the terminal object often prove to be very useful and often provide insight into the existence or non-existence of certain functors and natural transformations.

Chapter 3

Changing the Set of Labels

So far we have been considering the categories LRelSp and PMFrm of labelled-relational spaces and polymodal frames whose labelled components range over some fixed set I. In this chapter we turn our attention to the question of how spaces and frames labelled by elements one set can be altered so that the labels now belong to some different set. To fix some notation we write LRelSp/I and PMFrm/I to signify that the labelled components of the objects in each category are indexed by the set I.

This chapter contains the other two main results of the report — two new adjunctions are presented which give the relabelling of both categories.

3.1 Relabelling the Spaces

Suppose we have a function

$$r : I \longrightarrow J$$

between two sets I and J. This determines a covariant functor

$$\mathfrak{F} : \mathsf{LRelSp}/J \longrightarrow \mathsf{LRelSp}/I$$

which acts on objects as

$$(X, \tau, (\xrightarrow{j} \mid j \in J)) \longmapsto (X, \tau, (\xrightarrow{ri} \mid i \in I))$$

and on arrows as

$$f \longmapsto f.$$

To keep the results as general as possible, no restrictions are placed on the relabelling function. However, in Chapter 4, we shall see that if it is restricted in a certain way, then the adjunctions which relabel the spaces and the frames fit together very well with the adjunction of the previous chapter.

The next two subsections demonstrate that $\mathfrak{F}$ does not have a left adjoint, but does have a right adjoint.

3.1.1 Non-Existence of a Left Adjoint

The easiest way to see why there cannot be a left adjoint for $\mathfrak{F}$ is to consider the **terminal object** in LRelSp/J. We can give a snappy description of this as follows:

- The carrying set is $\mathcal{P}(J)$.
- The topology on $\mathcal{P}(J)$ is the indiscrete topology.
- For $X, Y \in \mathcal{P}(J)$, we have $X \xrightarrow{j} Y$ if and only if $j \in X$.

If $\mathfrak{F}$ were a right adjoint then it would preserve the terminal object. As, in general, we wish to consider arbitrary sets I and J and an arbitrary function $r : I \longrightarrow J$, $\mathfrak{F}$ cannot guarantee to preserve this. For example, if I and J have different cardinalities then, because $\mathfrak{F}$ preserves the carrying set of a space, the image of the terminal object from LRelSp/J is not the terminal object of LRelSp/I. Even if I and J do have the same size, unless we make some other restriction on r (such as being bijective, which is much too strong to be useful), $\mathfrak{F}$ will not preserve the terminal object.

3.1.2 Construction of the Right Adjoint

With the considerations of the terminal object in mind, it is clear that the object assignment of any possible right adjoint for $\mathfrak{F}$ must change the carrying set of the space. As such the flavour of the following construction is similar to that of the adjunction presented in Chapter 2. We first define a set of pre-points which will consist of certain pairs, then restrict this set in such a way that the remaining pre-points will be the elements of the new carrying set. The right adjoint to $\mathfrak{F}$ will be called $\mathfrak{G}$.

Definition. Let $(X, \tau, (\xrightarrow{i} \mid i \in I))$ be a labelled-relational space. A **pre-point** of $(X, \tau, (\xrightarrow{i} \mid i \in I))$ is a pair $(x, \mathcal{C})$ where $x \in X$ and $\mathcal{C} = \{C_j \mid j \in J\}$ is an indexed family of *closed* subsets of X, one for each element of J.

Again there is the notion of the j-**projection** of each pre-point which is the pair (x, C_j). Two j-projections are **related**, $(x, C_j)\ R_j\ (y, D_j)$ if and only if

$$\forall i \,.\, ri = j \,.\, x \xrightarrow{i} y \tag{3.1}$$

$$y \in C_j \tag{3.2}$$

Two pre-points are j-**related**, $(x, \mathcal{C}) \xrightarrow{j} (y, \mathcal{D})$, if and only if their j-projections are R_i related.

Definition. The carrying set $\mathbb{X}$, created by $\mathfrak{G}$, is the *largest* set Q of pre-points which satisfies the following two conditions:

$$(x, \mathcal{C}) \in Q \ \ \& \ \ C_j \cap U \neq \emptyset \Longrightarrow \exists (y, \mathcal{D}) \in Q.\, (x, \mathcal{C}) \xrightarrow{j} (y, \mathcal{D}) \ \ \& \ \ y \in U. \tag{3.3}$$

$$\forall i \in I \,.\, \forall y \in X \,.\, x \xrightarrow{i} y \Longrightarrow y \in C_{ri} \tag{3.4}$$

where $U \in \tau$.

When $j \in r(I)$ the conditions (3.3) and (3.4) capture the property that C_j is the closure of the set of i-related points over all the relations with $ri = j$.

To complete the object assignment we just need to give the new topology.

Definition. The **topology**, τ', on $\mathbb{X}$ is created from the topology τ on X as follows. We take each $U \in \tau$ and set $U' \in \tau'$ to be

$$U' = \{(x, \mathcal{C}) \in \mathbb{X} \,|\, x \in U\}. \tag{3.5}$$

Before embarking on the main result of this section we define the counit of the adjunction and show that its components are arrows in LRelSp/I.

Definition. The **counit** $\varepsilon_X : (\mathbb{X}, \tau', (\xrightarrow{ri} \,|\, i \in I)) \longrightarrow (X, \tau, (\xrightarrow{i} \,|\, i \in I))$ is given by

$$\varepsilon_X(x, \mathcal{C}) = x. \tag{3.6}$$

Lemma 15. *The components of ε are continuous p-morphisms in LRelSp/I.*

Proof. By the definition of the topology on $\mathbb{X}$ it is clear that the components of ε are continuous, therefore it remains to check (2.1) and (2.2). If $(x, \mathcal{C}) \xrightarrow{ri} (y, \mathcal{D})$ then by (3.1) $x \xrightarrow{i} y$. Finally, if $\varepsilon(x, \mathcal{C}) = x \xrightarrow{i} y \in U \in \tau$ then by (3.3) and (3.4) $y \in C_{ri} \cap U$ and so there is some $(x', \mathcal{C}')$ such that $(x, \mathcal{C}) \xrightarrow{i} (x', \mathcal{C}')$ and $x' \in U$. $\square$

We are now equipped with enough information to begin the proof of the adjunction which allows for the relabelling of the spaces. This is formalised as follows.

Theorem 2. *The map $(X, \tau, (\xrightarrow{i} \,|\, i \in I)) \longmapsto (\mathbb{X}, \tau', (\xrightarrow{j} \,|\, j \in J))$ is the object part of a covariant functor, $\mathfrak{G}$, which is right adjoint to $\mathfrak{F}$.*

Again the first step in the proof of this result is to define the arrows which arise from the universal property.

Definition. Let $(X, \tau, (\xrightarrow{j} \,|\, j \in J))$ be a labelled-relational space from LRelSp/J and let $(Y, \sigma, (\xrightarrow{i} \,|\, i \in I))$ be from LRelSp/I. Also let

$$f : (X, \tau, (\xrightarrow{ri} \,|\, i \in I)) \longrightarrow (Y, \sigma, (\xrightarrow{i} \,|\, i \in I))$$

be a continuous p-morphism.

Define $f^\sharp : (X, \tau, (\xrightarrow{j} \,|\, j \in J)) \longrightarrow (\mathbb{Y}, \sigma', (\xrightarrow{j} \,|\, j \in J))$ by

$$f^\sharp(x) = (fx, \{C_j \,|\, j \in J\}) \text{ where } C_j = \overline{\{y |\, fx \xrightarrow{i} y \ \ \& \ \ ri = j\}} \tag{3.7}$$

Certainly $f^\sharp(x)$ is a pre-point, so we have to show that it is also an element of $\mathbb{Y}$ and that $f^\sharp$ is indeed a continuous p-morphism. These results are the subjects of the following lemmas.

Lemma 16. *If $x \in X$ then $f^\sharp(x) \in \mathbb{Y}$.*

Proof. To prove this result, $f^\sharp(x)$ must satisfy (3.3) and (3.4). If $C_j \cap U \neq \emptyset$ then $\{y|\, fx \xrightarrow{i} y \ \ \& \ \ ri = j\} \cap U \neq \emptyset$. Therefore, as f is a continuous p-morphism there is x' such that $x \xrightarrow{i} x'$ and $fx' \in U$. As the relation $x \xrightarrow{i} x'$ holds in $\mathfrak{F}X$, all the relations $\xrightarrow{i}$ with $ri = j$ are the same and so we can conclude that $f^\sharp(x) \xrightarrow{j} (fx', \mathcal{D}) = f^\sharp(x')$. (3.4) is automatically fulfilled by the definition of $f^\sharp$. □

Lemma 17. *$f^\sharp$ is continuous.*

Proof. $x \in (f^\sharp)^{\leftarrow}(U')$ iff $f^\sharp(x) \in U'$ iff $f(x) \in U$ iff $x \in f^{\leftarrow}(U)$. As f is continuous, $f^{\leftarrow}(U)$ is open and therefore $(f^\sharp)^{\leftarrow}(U')$ is open too. □

Lemma 18. *If $x, y \in X$ are such that $x \xrightarrow{j} y$ then $f^\sharp(x) \xrightarrow{j} f^\sharp(y)$.*

Proof. Suppose $f^\sharp(x) = (fx, \{C_j|\, j \in J\})$. If $x \xrightarrow{j} y$ then, considering these relations in $\mathfrak{F}X$, we have that for all i such that $ri = j$, $x \xrightarrow{i} y$. Hence, as f is a continuous p-morphism, $fx \xrightarrow{i} fy$ for all i with $ri = j$ and so $fy \in C_j$ which means $f^\sharp(x) \xrightarrow{j} f^\sharp(y)$. □

Lemma 19. *If $x \in X$ and $f^\sharp(x) \xrightarrow{j} (y, \mathcal{D}) \in U' \in \tau'$, then there exists $z \in X$ such that $x \xrightarrow{j} z$ and $f^\sharp(z) \in U'$.*

Proof. If $x \in X$ and $f^\sharp(x) \xrightarrow{j} (y, \mathcal{D}) \in U'$ then, by the definition of $\xrightarrow{j}$, $y \in C_j$ and for each i with $ri = j$, $fx \xrightarrow{i} y$. Thus, for each i with $ri = j$, there exists some z_i such that $x \xrightarrow{i} z_i$. However, as these relations hold in $\mathfrak{F}X$, all the $\xrightarrow{i}$ with $ri = j$ are the same and so we may choose any of the z_i to be the required z in the statement of the lemma. □

Finally we need to show that $f^\sharp$ is the unique arrow with the required properties.

Lemma 20. *If $g : (X, \tau, (\xrightarrow{j} \,|\, j \in J)) \longrightarrow (\mathbb{Y}, \sigma', (\xrightarrow{j} \,|\, j \in J))$ satisfies the condition $\varepsilon_Y \circ \mathfrak{F}g = f$ then $g = f^\sharp$.*

Proof. Suppose $f^\sharp(x) = (fx, \mathcal{C})$ and $g(x) = (\xi, \Gamma)$. As ε_Y just returns the first component of the pair, by the property of g mentioned above, we must have that $\xi = fx$.

Now, by (3.4), $\{z|\, \xi \xrightarrow{i} z \ \ \& \ \ ri = j\} \subseteq \gamma_j$ and so $\overline{\{z|\, \xi \xrightarrow{i} z \ \ \& \ \ ri = j\}} \subseteq \overline{\gamma_j} = \gamma_j$ since γ_j is closed. Thus $C_j \subseteq \gamma_j$.

Conversely, if $\gamma_j \nsubseteq C_j$, then $\gamma_j \cap (X \setminus C_j) \neq \emptyset$ and $(X \setminus C_j)$ is open. As this intersection is non-empty, there exists some (ζ, Δ) such that $(\xi, \Gamma) \xrightarrow{j} (\zeta, \Delta)$, but this means that $\xi \xrightarrow{i} \zeta$ for all i with $ri = j$, and $\zeta \in (X \setminus C_j)$. This is a contradiction since, by definition, $\zeta \in C_j$. Hence $\gamma_j \subseteq C_j$. □

This completes the proof of Theorem 2, so we have shown that $\mathfrak{F} \dashv \mathfrak{G}$. To finish this section it is worth recording explicitly the arrow assignment of $\mathfrak{G}$ and the unit of the adjunction.

Corollary 21. *The functor* $\mathfrak{G} : \mathsf{LRelSp}/I \longrightarrow \mathsf{LRelSp}/J$ *acts on arrows* $f : X \longrightarrow Y$ *as*

$$\mathfrak{G}(f)(x, \mathcal{C}) = \left(fx\,,\ \overline{\{y \mid fx \xrightarrow{i} y\ \ \&\ \ ri = j\}} \right).$$

The unit, η*, of the adjunction is given by*

$$\eta_A(x) = (x, \{C_j \mid j \in J\}) \quad \textit{where} \quad C_j = \overline{\{y \mid x \xrightarrow{j} y\}}. \tag{3.8}$$

3.2 Relabelling the Frames

The relabelling function $r : I \longrightarrow J$, between the two sets I and J from the previous section also determines an analogue of the functor $\mathfrak{F}$, this time between the algebraic categories. We define

$$\mathcal{F} : \mathsf{PMFrm}/J \longrightarrow \mathsf{PMFrm}/I$$

which acts on objects as

$$(A, \leqslant, ([j], \langle j \rangle \mid j \in J)) \longmapsto (A, \leqslant, ([ri], \langle ri \rangle \mid i \in I))$$

and on arrows as

$$f \longmapsto f.$$

In this section we shall construct a right adjoint for $\mathcal{F}$ and give this adjunction. In contrast to the two previous adjunctions, this one is best described by beginning with an explicit definition of the right adjoint.

The functor

$$\mathcal{G} : \mathsf{PMFrm}/I \longrightarrow \mathsf{PMFrm}/J,$$

which is right adjoint to $\mathcal{F}$ is defined to act on objects as

$$(A, \leqslant, ([i], \langle i \rangle \mid i \in I)) \longmapsto (A, \leqslant, ([j]', \langle j \rangle' \mid j \in J))$$

and on arrows as

$$f \longmapsto f.$$

The box and diamond operations are given by

$$[j]'a = \bigwedge_i \{[i]\,a \mid ri = j\} \tag{3.9}$$

$$\langle j \rangle' a = \bigwedge_i \{\langle i \rangle\,a \mid ri = j\} \tag{3.10}$$

with the exception that $\langle j \rangle' \bot = \bot$ when $j \notin r(I)$. The next two lemmas will show that $\mathcal{G}$ is a (covariant) functor.

Lemma 22. *The functions* $[j]'$ *and* $\langle j\rangle'$ *are monotone.*

Proof. If $a \leqslant b$ then, for all labels $i \in I$, $[i]\, a \leqslant [i]\, b$ and so $[j]'a \leqslant [j]'b$. Similarly for $\langle j\rangle'$. $\square$

Lemma 23. *The functions* $[j]'$ *and* $\langle j\rangle'$ *satisfy the conditions of* (2.4).

Proof. For all labels i, $\top \leqslant [i]\,\top$ and so certainly $\top \leqslant [j]'\top$.

$$\begin{aligned}
[j]'a \wedge [j]'b &= \bigwedge_i \{[i]a \mid ri = j\} \wedge \bigwedge_k \{[k]b \mid rk = j\} \\
&= \bigwedge_i \{[i]a \wedge \bigwedge_k \{[k]b \mid rk = j\} \mid ri = j\} \\
&= \bigwedge_{i,k} \{[i]a \wedge [k]b \mid ri = rk = j\} \\
&\leqslant \bigwedge_i \{[i]a \wedge [i]b \mid ri = j\} \\
&\leqslant \bigwedge_i \{[i](a \wedge b) \mid ri = j\} \\
&= [j]'(a \wedge b).
\end{aligned}$$

$$\begin{aligned}
\langle j\rangle'a \wedge [j]'b &= \bigwedge_i \{\langle i\rangle a \mid ri = j\} \wedge \bigwedge_k \{[k]b \mid rk = j\} \\
&= \bigwedge_i \{\langle i\rangle a \wedge \bigwedge_k \{[k]b \mid rk = j\} \mid ri = j\} \\
&= \bigwedge_{i,k} \{\langle i\rangle a \wedge [k]b \mid ri = rk = j\} \\
&\leqslant \bigwedge_i \{\langle i\rangle a \wedge [i]b \mid ri = j\} \\
&\leqslant \bigwedge_i \{\langle i\rangle (a \wedge b) \mid ri = j\} \\
&= \langle j\rangle'(a \wedge b).
\end{aligned}$$

Finally, since $\langle i\rangle\bot \leqslant \bot$ for each i, we have $\langle j\rangle'\bot \leqslant \bot$. $\square$

With the knowledge that $\mathcal{G}$ really is a functor we proceed to the relabelling of the frames, which is formalised as follows.

Theorem 3. *There is a covariant adjunction between* PMFrm/J *and* PMFrm/I *with* $\mathcal{G}$ *right adjoint to* $\mathcal{F}$.

To prove this we shall exhibit the unit, $\eta : 1 \longrightarrow \mathcal{GF}$, and counit, $\varepsilon : \mathcal{FG} \longrightarrow 1$, of the adjunction and show that these are natural transformations which satisfy the required identities.

Definition. The underlying functions for the **unit**

$$\eta_A : (A, \leqslant, ([j], \langle j\rangle \mid j \in J)) \longrightarrow (A, \leqslant, ([j]', \langle j\rangle' \mid j \in J))$$

and **counit**

$$\varepsilon_A : (A, \leqslant, ([ri]', \langle ri\rangle' \mid i \in I)) \longrightarrow (A, \leqslant, ([i], \langle i\rangle \mid i \in I))$$

are both given by the identity map $a \longmapsto a$.

Lemma 24. *The components of η are arrows in* PMFrm/J.

Proof. η_A is clearly a frame morphism, so we need only check conditions (2.5) and (2.6), i.e. that $[j]\,a \leqslant [j]'a$ and $\langle j\rangle a \leqslant \langle j\rangle' a$ for all $a \in A$. Both of these follow immediately, because for $j \in r(I)$, $[j] = [j]'$ and $\langle j\rangle = \langle j\rangle'$. $\square$

Lemma 25. *The components of ε are arrows in* PMFrm/I.

Proof. Similarly, ε_A is obviously a frame morphism, so we just have to check (2.5) and (2.6) i.e. that $[ri]'a \leqslant [i]\,a$ and $\langle ri\rangle a \leqslant \langle i\rangle a$ for all $a \in A$. These again follow immediately, by the definitions of $[ri]'$ and $\langle ri\rangle'$ as meets. $\square$

Lemma 26. *η and ε are natural transformations.*

Proof. This follows because the underlying functions of the unit and counit are both the identity map and so the two diagrams below easily commute.

$$\begin{tikzcd}
(A, \leqslant, ([j], \langle j\rangle \mid j \in J)) \arrow[r, "\eta_A"] \arrow[d, "f"'] & (A, \leqslant, ([j]', \langle j\rangle' \mid j \in J)) \arrow[d, "f"] \\
(B, \preccurlyeq, ([j], \langle j\rangle \mid j \in J)) \arrow[r, "\eta_B"] & (B, \preccurlyeq, ([j]', \langle j\rangle' \mid j \in J))
\end{tikzcd}$$

$$\begin{tikzcd}
(A, \leqslant, ([ri]', \langle ri\rangle' \mid i \in I)) \arrow[r, "\varepsilon_A"] \arrow[d, "f"'] & (A, \leqslant, ([i], \langle i\rangle \mid i \in I)) \arrow[d, "f"] \\
(B, \preccurlyeq, ([ri]', \langle ri\rangle' \mid i \in I)) \arrow[r, "\varepsilon_B"] & (B, \preccurlyeq, ([i], \langle i\rangle \mid i \in I))
\end{tikzcd}$$

$\square$

Lemma 27. *η and ε satisfy the triangle identities:*

$$
\begin{tikzcd}
\mathcal{F}A \arrow[r, "\mathcal{F}(\eta_A)"] \arrow[dr, "1_{\mathcal{F}A}"'] & \mathcal{F}\mathcal{G}\mathcal{F}A \arrow[d, "\varepsilon_{\mathcal{F}A}"] \\
& \mathcal{F}A
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{G}\mathcal{F}\mathcal{G}B \arrow[d, "\mathcal{G}(\varepsilon_B)"'] & \mathcal{G}B \arrow[l, "\eta_{\mathcal{G}B}"'] \arrow[dl, "1_{\mathcal{G}B}"] \\
\mathcal{G}B &
\end{tikzcd}
$$

Proof. Suppose A and B are polymodal frames from the appropriate categories, then this result follows almost immediately, because $\varepsilon_{\mathcal{F}A}$ and $\mathcal{F}(\eta_A)$ are both the identity arrow of $\mathcal{F}A$. Similarly, $\mathcal{G}(\varepsilon_B)$ and $\eta_{\mathcal{G}B}$ are both the identity arrow of $\mathcal{G}B$. $\square$

This brings us to the end of the proof of the covariant adjunction $\mathcal{F} \dashv \mathcal{G}$.

Chapter 4

Composition

Armed with the battery of results from the previous two chapters, it is now natural to ask how they fit together. In this chapter we shall consider how the various gadgets which have been introduced compose. We begin by considering how the re-labelling functors $\mathfrak{F}$ and $\mathcal{F}$ of Chapter 3 compose with the functors $\mathcal{O}$ and $\mathcal{T}$ from Chapter 2, then proceed to see how the adjunctions themselves fit together. Finally we shall consider how two different relabelling functions, $r : I \longrightarrow J$ and $s : K \longrightarrow J$, with a common codomain interact with each other by considering whether or not the appropriate Beck-Chevalley Conditions hold.

4.1 The Functors

In this section we are concerned with the interaction between the functors $\mathfrak{F}$ and $\mathcal{F}$ which do the relabelling and the functors $\mathcal{O}$ and $\mathcal{T}$ which provide the transition between labelled-relational spaces and polymodal frames. In particular, we ask whether or not either of the following two squares commute.

$$
\begin{tikzcd}
\mathsf{LRelSp}/J \arrow[r, "\mathfrak{F}"] \arrow[d, "\mathcal{O}"'] & \mathsf{LRelSp}/I \arrow[d, "\mathcal{O}'"] \\
\mathsf{PMFrm}/J \arrow[r, "\mathcal{F}"] & \mathsf{PMFrm}/I
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{LRelSp}/J \arrow[r, "\mathfrak{F}"] & \mathsf{LRelSp}/I \\
\mathsf{PMFrm}/J \arrow[r, "\mathcal{F}"] \arrow[u, "\mathcal{T}"] & \mathsf{PMFrm}/I \arrow[u, "\mathcal{T}'"']
\end{tikzcd}
$$

It turns out that the diagram on the left does commute. The reasons for this are that $\mathfrak{F}$ does not change the topology of the space, therefore the underlying frame produced by $\mathcal{O}$ and $\mathcal{O}'$ is the same. Also, as both $\mathfrak{F}$ and $\mathcal{F}$ essentially 'copy across' the labelled components, the box and diamond operations on the resulting polymodal frame in PMFrm/I are induced from the same relations.

Sadly, life is not so simple when it comes to the diagram on the right. This square does not commute. The reason for this is similar to the explanation as to why $\mathfrak{F}$ does not have a left adjoint. To see why this diagram does not commute, it is worth recalling how polymodal frame points are created. Remember that the second component of each polymodal frame point is an indexed family of elements of the frame, one for each member of the appropriate labelling set. These families can therefore be thought of as (possibly infinite) tuples, indeed in the example below we shall display them as such. If I and J have different cardinalities, then the tuples created by $\mathcal{T}$ and $\mathcal{T}'$ will be of different lengths. Thus the carrying set created by $\mathcal{T}$ will be different from that created by $\mathcal{T}' \circ \mathcal{F}$. Even if I and J do have the same cardinality, it is possible to define a polymodal frame in PMFrm/J and a re-labelling function $r : I \longrightarrow J$ such that when applied to this frame $\mathfrak{F} \circ \mathcal{T} \neq \mathcal{T}' \circ \mathcal{F}$. Moreover, there is not even a natural isomorphism between the images.

Example 1. Let $\mathbf{3}$ be the 3-element frame $\{\bot \leqslant \star \leqslant \top\}$ and consider the polymodal frame $(\mathbf{3}, \leqslant, [a], \langle a \rangle, [b], \langle b \rangle)$ from the category $\mathsf{PMFrm}/\{a, b\}$ where

$$
\begin{aligned}
[a]x &= x & \qquad [b]x &= x \\
\langle a \rangle x &= x & \qquad \langle b \rangle \top &= \top \\
& & \langle b \rangle \star &= \top \\
& & \langle b \rangle \bot &= \bot
\end{aligned}
$$

and the re-labelling function $r : \{\alpha, \beta\} \longrightarrow \{a, b\}$ is given by $r(x) = b$.

In this example $I = \{\alpha, \beta\}$ and $J = \{a, b\}$. For this frame, the two possible frame points are determined by their behaviour on the element $\star$, so let $p_\top$ denote the frame point with $p_\top(\star) = \bot$ and $p_\star$ denote the frame point with $p_\star(\star) = \top$.

When $\mathcal{T}$ is applied to this frame it produces three polymodal frame points, namely $(p_\top, \star, \bot)$, $(p_\top, \bot, \bot)$ and $(p_\star, \bot, \bot)$. However, when $\mathcal{T}' \circ \mathcal{F}$ is applied to it there are only two polymodal frame points, this time just $(p_\top, \bot, \bot)$ and $(p_\star, \bot, \bot)$.

(Using the idea of viewing the indexed family in each polymodal frame point as a tuple, in each of these triples the first component is the frame point, the middle component is the element corresponding to the label α and the last component is the element corresponding to the label β.)

This shows that r must be at least injective (and so bijective for finite I and J) if there is to be any chance of the square commuting. This requirement is again too strong to be worth adopting.

4.2 The Adjunctions

In this section we consider how the adjunctions from the previous chapters fit together. The situation can be pictured as follows.

$$\begin{tikzcd}
\mathsf{LRelSp}/J \arrow[r, shift left] \arrow[d, leftrightarrow] & \mathsf{LRelSp}/I \arrow[l, shift left] \arrow[d, leftrightarrow] \\
\mathsf{PMFrm}/J \arrow[r, shift left] & \mathsf{PMFrm}/I \arrow[l, shift left]
\end{tikzcd}$$

It is well known that adjunctions compose, but to do this we need to be sure which functor in each adjunction is the left adjoint and which is the right adjoint. It is therefore worth untwisting the contravariance of the vertical sides in the square above. We achieve this by taking the opposites of the algebraic categories. This yields the following square. (The subscripts are used merely to tag which functors are the left and right adjoints.)

$$\begin{tikzcd}
\mathsf{LRelSp}/J \arrow[r, shift left, "\mathfrak{F}_l"] \arrow[d, shift left, "\mathcal{O}_l"] & \mathsf{LRelSp}/I \arrow[l, shift left, "\mathfrak{G}_r"] \arrow[d, shift left, "\mathcal{O}'_l"] \\
(\mathsf{PMFrm}/J)^{\mathrm{op}} \arrow[r, shift left, "\mathcal{F}_r"] \arrow[u, shift left, "\mathcal{T}_r"] & (\mathsf{PMFrm}/I)^{\mathrm{op}} \arrow[l, shift left, "\mathcal{G}_l"] \arrow[u, shift left, "\mathcal{T}'_r"]
\end{tikzcd} \tag{4.1}$$

Note that in doing this the components of the adjunction between the (opposites of the) categories of polymodal frames switch their rôles as left and right adjoints.

The only way to compose the sides of this square so that the composition will also be an adjunction is to compose the top, right and bottom sides to give $\mathcal{G} \circ \mathcal{O}' \circ \mathfrak{F} \dashv \mathfrak{G} \circ \mathcal{T}' \circ \mathcal{F}$. So how does this new adjunction compare with $\mathcal{O} \dashv \mathcal{T}$ which constitutes the left side of the square? The easiest way to answer this question is to see what happens when $\mathcal{G} \circ \mathcal{O}' \circ \mathfrak{F}$ is applied to an object from LRelSp/J and compare this with the result of applying just $\mathcal{O}$ to the same object. Given $(X, \tau, (\xrightarrow{j} \mid j \in J))$ the composite path gives

$$\begin{tikzcd}
(X, \tau, (\xrightarrow{j} \mid j \in J)) \arrow[r, "\mathfrak{F}"] & (X, \tau, (\xrightarrow{ri} \mid i \in I)) \arrow[d, "\mathcal{O}'"] \\
(\tau, \subseteq, (\llbracket j \rrbracket', \langle j \rangle' \mid j \in J)) & (\tau, \subseteq, (\llbracket ri \rrbracket, \langle ri \rangle \mid i \in I)) \arrow[l, "\mathcal{G}"']
\end{tikzcd}$$

and the image under $\mathcal{O}$ is just the usual $(\tau, \subseteq, ([j], \langle j\rangle \mid j \in J))$. Clearly the underlying frame is the same, but what about the boxes and diamonds? Do we have $[j]' = [j]$ and $\langle j\rangle' = \langle j\rangle$? The situation is (almost[1]) the same for both, so we need only consider the boxes.

If $j \in r(I)$ then $[j] = [j]'$, because when $\mathcal{G}$ takes meets to form each new box all the $[i]$ with $ri = j$ are the same. This, in turn, is due to the copying nature of $\mathfrak{F}$. However, when $j \notin r(I)$ the boxes created by $\mathcal{O}$ and by $\mathcal{G} \circ \mathcal{O}' \circ \mathfrak{F}$ are different. The reason for this is that $\mathfrak{F}$ loses all of the information about the relations whose labels are elements of J that are not in the image of r. Thus when $\mathcal{G}$ creates those boxes whose labels are not in the image of r it takes the default option of giving the empty meet.

In the opening section of the previous chapter I said that if we restricted r in some way then the adjunctions would fit together particularly well. It should now be clear what this restriction is.

Proposition 28. *If r is surjective then both $\mathcal{O}$ and $\mathfrak{G} \circ \mathcal{O}' \circ \mathcal{F}$ are the same functor, with right adjoint given by either $\mathcal{T}$ or $\mathfrak{G} \circ \mathcal{T}' \circ \mathcal{F}$.*

Proof. The discussion above should suffice as justification and the last statement follows because any two right adjoints to a given functor are naturally isomorphic. ☐

Thus, when r is surjective, the adjunctions forming the square (4.1) commute exactly in the only way they can.

4.3 Beck-Chevalley Conditions

Suppose now that we are given two relabelling functions $r : I \longrightarrow J$ and $s : K \longrightarrow J$. Viewing this situation in Set we can form the pullback of these two arrows as follows:

$$\begin{tikzcd} L \arrow[r, "\pi'"] \arrow[d, "\pi"'] & K \arrow[d, "s"] \\ I \arrow[r, "r"'] & J \end{tikzcd} \tag{4.2}$$

where $L = \{(i, k) \in I \times K \mid r(i) = s(k)\}$ and π, π' are the obvious projections.

The functions in this diagram can be used to give the relabelling which underlies each of the functors from Chapter 3. Therefore it is worth asking if this pullback lifts to a certain commuting square of functors between the spatial or between the algebraic categories.

[1]Remember the minor exception that when $j \notin r(I)$, the diamond operations created by $\mathcal{G}$ satisfy $\langle j\rangle' \bot = \bot$.

4.3.1 The Spatial Categories

The question here is whether or not the following square commutes:

$$\begin{tikzcd}
\mathsf{LRelSp}/L \arrow[r, "\mathfrak{G}_{\pi'}"] & \mathsf{LRelSp}/K \\
\mathsf{LRelSp}/I \arrow[u, "\mathfrak{F}_{\pi}"] \arrow[r, "\mathfrak{G}_{r}"] & \mathsf{LRelSp}/J \arrow[u, "\mathfrak{F}_{s}"']
\end{tikzcd} \tag{4.3}$$

In general this diagram does not commute, even up to isomorphism and it is easy to give an example to illustrate this.

Example 2. Let the sets of labels be

$$I = \{1\} \qquad J = \{1, 2\} \qquad K = \{1, 2, 3\}$$

and let the relabelling functions, $r : I \longrightarrow J$ and $s : K \longrightarrow J$, be given by

$$r(1) = 1 \qquad s(1) = 1 \qquad s(2) = s(3) = 2.$$

This means that L is just the singleton set $\{(1, 1)\}$.

Now consider what happens to the terminal object in LRelSp/I as it is mapped along the two paths around the square (4.3). First the path $\mathfrak{F}_s \circ \mathfrak{G}_r$. As $\mathfrak{G}_r$ is a right adjoint it preserves the terminal object and so, as $\mathfrak{F}_s$ does not change the underlying topological space, the carrying set produced under $\mathfrak{F}_s \circ \mathfrak{G}_r$ has $4 = |\mathcal{P}(J)|$ elements. Now for the path $\mathfrak{G}_{\pi'} \circ \mathfrak{F}_\pi$. In this case $\mathfrak{F}_\pi$ is just the identity functor, so applying $\mathfrak{G}_{\pi'} \circ \mathfrak{F}_\pi$ to the terminal object of LRelSp/I yields the terminal object in LRelSp/K, the carrying set of which has $8 = |\mathcal{P}(K)|$ elements. Thus there is no hope of an isomorphism.

However, we can say something about the functors which form the two paths around the square (4.3). Under $\mathfrak{F}_s \circ \mathfrak{G}_r$ the carrying set of the image of each object from LRelSp/I consists of certain pairs $(x, \{C_j \mid j \in J\})$, whereas under $\mathfrak{G}_{\pi'} \circ \mathfrak{F}_\pi$ the pairs are of the form $(x, \{C_k \mid k \in K\})$. This means that there is some redundancy in the points of the space formed under $\mathfrak{F}_s \circ \mathfrak{G}_r$. The next two propositions show that there is a natural transformation between the composite functors which form the two paths around (4.3) in one direction, but not in the other.

Proposition 29. *The map $\theta : \mathfrak{F}_s \circ \mathfrak{G}_r \longrightarrow \mathfrak{G}_{\pi'} \circ \mathfrak{F}_\pi$ given by*

$$(x, \{C_j \mid j \in J\}) \longmapsto (x, \{C_{s(k)} \mid k \in K\})$$

on the carrying set of the space defines a natural transformation.

Proof. A diagram chase around (4.3) shows that those closed sets indexed by elements of J that are not in the image of s play no part in defining the relations that hold in $\mathsf{LRelSp}/\mathrm{K}$ under $\mathfrak{F}_s \circ \mathfrak{G}_r$. Also, the indexed families are not used in the definition of the topology of the space created under either path around the square. Thus in removing those closed sets that are indexed by $j \notin s(K)$ the only information that is lost would not have been used anyway. To ensure that the following square commutes,

$$\begin{array}{ccc} (\mathfrak{F}_s \circ \mathfrak{G}_r)X & \xrightarrow{\theta_X} & (\mathfrak{G}_{\pi'} \circ \mathfrak{F}_\pi)X \\ \Big\downarrow f & & \Big\downarrow f \\ (\mathfrak{F}_s \circ \mathfrak{G}_r)Y & \xrightarrow{\theta_Y} & (\mathfrak{G}_{\pi'} \circ \mathfrak{F}_\pi)Y \end{array}$$

we need only observe that θ is essentially just an identity map with the additional property that it removes the unnecessary facilities of the points. ☐

Proposition 30. *There is no natural transformation in the other direction, from $\mathfrak{G}_{\pi'} \circ \mathfrak{F}_\pi$ to $\mathfrak{F}_s \circ \mathfrak{G}_r$.*

Proof. This follows from a further consideration of Example 2 above. Given the terminal object T in LRelSp/I, its image under $\mathfrak{G}_{\pi'} \circ \mathfrak{F}_\pi$ contains a point, x, that is related to all the other points (and itself) by $\xrightarrow{2}$ and is not related to anything by any other relation. However, in the image of T under $\mathfrak{F}_s \circ \mathfrak{G}_r$ any point that is related to others by $\xrightarrow{2}$ is also related to them by $\xrightarrow{3}$. Condition (2.2) therefore proves that there is no such natural transformation from $\mathfrak{G}_{\pi'} \circ \mathfrak{F}_\pi$ to $\mathfrak{F}_s \circ \mathfrak{G}_r$. For, suppose ν were such a transformation, then because ν preserves the relations, $\nu(x) \xrightarrow{2} \nu(y)$ and so $\nu(x) \xrightarrow{3} \nu(y)$ for all $y \in (\mathfrak{G}_{\pi'} \circ \mathfrak{F}_\pi)T$. As ν is a continuous p-morphism this means there is some x' in $(\mathfrak{G}_{\pi'} \circ \mathfrak{F}_\pi)T$ such that $x \xrightarrow{3} x'$, but this is a contradiction. ☐

4.3.2 The Algebraic Categories

The diagram for the categories of polymodal frames is as follows.

$$\begin{array}{ccc} \mathsf{PMFrm}/L & \xrightarrow{\mathcal{G}_{\pi'}} & \mathsf{PMFrm}/K \\ \Big\uparrow \mathcal{F}_\pi & & \Big\uparrow \mathcal{F}_s \\ \mathsf{PMFrm}/I & \xrightarrow{\mathcal{G}_r} & \mathsf{PMFrm}/J \end{array} \tag{4.4}$$

Proposition 31. *This square commutes exactly.*

Proof. Let $(A, \leqslant, ([i], \langle i \rangle | \, i \in I))$ be an object from PMFrm/I. We shall see what happens to this object as it mapped along both paths around the square to PMFrm/K. As both $\mathcal{F}$ and $\mathcal{G}$ preserve the underlying frame, our only concern will be the boxes and diamonds. Again the situation is (almost[2]) the same for both, so we shall only consider the boxes.

Given $k \in K$ there are two cases to be worked through — either there is some $i \in I$ such that $r(i) = s(k)$ and so $(i, k) \in L$ or k does not form part of a pair in L. Suppose that $a \in A$, then in the first case the path $\mathcal{F}_s \circ \mathcal{G}_r$ gives

$$\begin{aligned}[k]a &= [sk]a \\ &= \bigwedge_i \{[i]a | \, r(i) = s(k)\}.\end{aligned}$$

The other path, $\mathcal{G}_{\pi'} \circ \mathcal{F}_\pi$, gives

$$\begin{aligned}[k]a &= \bigwedge_{(i,k)} \{[(i,k)]a | \, \pi'(i,k) = k\} \\ &= \bigwedge_i \{[i]a | \, r(i) = s(k)\}\end{aligned}$$

and so the result is clearly the same along either path.

In the second case, where k is such that there is no $i \in I$ such that $r(i) = s(k)$, we have that $s(k) \notin r(I)$ and so along the path $\mathcal{F}_s \circ \mathcal{G}_r$, $[k]$ gives the default value of the empty meet to every $a \in A$. Along the other path, $\mathcal{G}_{\pi'} \circ \mathcal{F}_\pi$, the fact that k is not part of a pair in L means that k is not in the image of π' and so $[k]$ again gives to each $a \in A$ the default value of the empty meet. $\square$

This shows that the pullback (4.2) lifts to a square of functors (4.4) between the categories of polymodal frames which commutes exactly.

[2]Same exception as mentioned in the previous footnote.

Chapter 5

Final Thoughts

The first adjunction (and its associated duality) of the report, presented in Chapter 2, provides a useful extension of the new approach to modal logic which began with [2]. The fact that there is a fairly straightforward generalisation to polymodal languages means that there are now semantic models for a far wider class of modal logics.

In the conclusions of [2], Hilken mentions two important potential applications to theoretical computer science: temporal logic based on the real numbers and dynamic logic based on Scott domains. He writes that,

> ... the interpretation of propositions as opens sets enforces a natural "observability" condition on the logic. In the case of temporal logic, it means that any proposition true at time t must be true over some open interval $(t-\delta, t+\delta)$ containing t. In the case of dynamic logic, it means that any proposition true of an infinite computation must become true at some finite stage in that computation. This combination of modal logic with observability conditions has great potential for application to domain theory, concurrency theory, and many other areas of computer science.

There are several areas in which further work could proceed from here. At the moment the sets of labels themselves have no internal structure and the labelled components operate independently and in parallel. However, if the labelling sets did have some structure of their own, then the labelled facilities could be forced to interact with each other in potentially interesting ways. The re-indexing adjunctions presented here provide an important first step towards developing a full dynamic logic based on this style of modal logic. With this view, distinct labels can be used to represent distinct programs in an abstract model of computation in which the points of the labelled-relational space correspond to possible states of the machine. An extension in this direction would be to find out if the adjunctions of Chapter 3 can be generalised to the case where the labelling sets have a simple programming language structure. This would develop rich models in which, on the spatial side, there may be relations which include compositions or iterations of others.

Another possibility is to consider the consequences of equipping the labelling sets with a partial order. This way of structuring the labels would lead towards a more domain theoretic approach, which would be potentially applicable to the semantics of programming. This could then be extended further to the cases where the index sets are (complete) lattices or quantales.

Finally, further work could look at other ways to index the labelled components and see whether or not the indexing can be internalised to the appropriate categories. The techniques described briefly in Section 1.2 might prove useful when the index sets are equipped with some of the structures mentioned above.

It is clear that the details of the relabelling constructions will have to develop to accommodate the additional structure placed on the labels, but the adjunctions presented in Chapter 3 provide a solid basis for the possible extensions mentioned above and for other further work in this area.

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