Semi-analytic Tableaux For Propositional Modal Logics of Nonmonotonicity

Rajeev Goré *
Department of Computer Science
University of Manchester
Manchester, M13 9PL
England

E-mail: rpg@cs.man.ac.uk

Abstract

The propositional monotonic modal logics **K45**, **K45D**, **S4.2**, **S4R** and **S4F** elegantly capture the semantics of many current *nonmonotonic* formalisms as long as (strong) deducibility of A from a theory Γ, $\Gamma \vdash A$, allows the use of necessitation on the members of Γ. This is usually forbidden in modal logic where Γ is required to be empty, resulting in a weaker notion of deducibility.

Recently, Marek, Schwarz and Truszcziński have given algorithms to compute the stable expansions of a *finite* theory Γ in various such nonmonotonic formalisms. Their algorithms assume the existence of procedures for deciding (strong) deducibility in these monotonic modal logics and consequently such decision procedures are important for automating nonmonotonic deduction.

We first give a sound, (weakly) complete and cut-free, semi-analytic tableau calculus for monotonic **S4R**, thus extending the cut elimination results of Schwarz for monotonic **K45** and **K45D**. We then give sound and complete semi-analytic tableau calculi for monotonic **K45**, **K45D**, **S4.2** and **S4F** by adding an (analytic) cut rule. The proofs of tableau completeness yield a deterministic satisfiability test to determine theoremhood (weak deducibility), $\vdash_L A$, because all proofs are constructive. The techniques are due to Hintikka and Rautenberg. We then show that the tableau calculi extend trivially to handle (strong) deducibility, $\Gamma \vdash A$, for *finite* Γ.

Keywords: modal theorem proving, semi-analytic tableaux, nonmonotonic modal logic.

*Research supported by the U.K. Science and Engineering Research Council under grant number GR/H/18449.

1 Introduction

Propositional modal logics have been used to model epistemic notions like knowledge and belief for quite a while now where the formula $\Box A$ is read as "A is believed" or as "A is known". Given a (monotonic) modal logic **S** and a set of formulae Γ, the formula A is a monotonic consequence of Γ in **S** if it is deducible in **S** from Γ, usually written as $\Gamma \vdash_S A$. The set Γ is usually called a theory and the monotonic consequences of Γ in **S** are all the formulae deducible from Γ in **S**; that is $Cn_S(\Gamma) = \{A \mid \Gamma \vdash_S A\}$. The system is "monotonic" in that if A is in $Cn_S(\Gamma)$ then it will be in $Cn_S(\Gamma')$ for any superset Γ' of Γ.

To obtain nonmonotonicity we assume $\neg\Box A$ ("A is not known") if there is no deduction of A in **S** from Γ and previous assumptions. More formally, the theory T is an **S**-expansion of theory Γ if it satisfies the equation $T = Cn_S(\Gamma \cup \{\neg\Box A \mid A \notin T\})$. Since T appears in the right hand side, the definition is circular, and consequently, a theory Γ may have zero, one, or more **S**-expansions. To compensate for this phenomenon, the set of *nonmonotonic* consequences of Γ in **S** is usually defined as the intersection of all **S**-expansions of Γ. The new system is "nonmonotonic" because, although A may be a nonmonotonic consequence of Γ, it may not be a nonmonotonic consequence of a superset of Γ.

Concurrently, various nonmodal formalisms have also been used to model epistemic notions giving rise to default logics and autoepistemic logics. Recently, the nonmonotonic modal logics based on the (monotonic) modal logics **K45**, **K45D**, **S4R**, **S4.2** and **S4F** have been shown to capture the minimal model semantics for some of these *nonmodal* nonmonotonic formalisms; for example, nonmonotonic **K45D** captures the semantics of autoepistemic logic while nonmonotonic **S4F** "naturally generalises default logic and autoepistemic logic" [Sch, Trua, Trub, ST]. Indeed, Marek, Schwarz and Truszczyński [MST91] give algorithms to compute the **S**-expansions of a *finite* theory Γ in a wide class of nonmonotonic modal logics. However their algorithms assume the existence of effective procedures for deciding deducibility, $\Gamma \vdash_S A$, in the underlying (monotonic) modal logic **S**. Therefore decision procedures for deducibility in these particular (monotonic) modal logics are crucial for automating nonmonotonic deduction in both *modal and nonmodal* formulations.

Using the technique of semi-analytic tableaux we provide deterministic and nondeterministic decision procedures for deducibility in monotonic **K45**, **K45D**, **S4R**, **S4.2** and **S4F**. Specifically, we give sound and complete (nondeterministic) tableau systems for deciding *theoremhood* in each of these (monotonic) modal logics. Each proof of tableau completeness is constructive thereby yielding a *deterministic* test for satisfiability and hence a *deterministic* test for theoremhood. The system for **S4R** does not have the subformula property but is cut-free proving that Gentzen's cut-elimination theorem holds for **S4R**. The systems for **S4.2** and **S4F** also break the subformula property and require an analytic cut rule for completeness but remain decidable. Cut-free systems for monotonic **K45** and **K45D** have been given already by Schwarz [Shv89] in sequent form, but by adding an analytic cut rule we obtain greatly simplified completeness proofs. The techniques are due to Hintikka and Rautenberg [Rau83, Rau85].

In (monotonic) modal logic, the ability to decide *theoremhood* in a logic **L** *does not* automatically enable us to decide (strong) deducibility, $\Gamma \vdash_L A$, in **L** because theoremhood is defined only when Γ is empty. The tableau calculi can be extended to handle deducibility, in a trivial way, as long as Γ is finite.

2 Definitions and Notational Conventions

2.1 Propositional Normal Modal Logics

We consider only propositional modal logics. We use a denumerable set of primitive propositions $\mathcal{P} = \{p_1, p_2, \cdots\}$ and use $\wedge, \neg$ and $\Box$ as primitives. Then the other usual connectives are defined as abbreviations: $(A \vee B) = (\neg(\neg A \wedge \neg B)); (A \Rightarrow B) = (\neg(A \wedge \neg B))$; and $(\Diamond A) = (\neg\Box\neg A)$ where the $=$ sign is merely a meta-linguistic notation. We also use 0 to denote a constant false proposition and use $\emptyset$ to denote the empty set.

The definition of (well formed) formulae is as usual. Lower case letters like p and q denote members of $\mathcal{P}$. Upper case letters from the beginning of the alphabet like A and B together with P and Q (all possibly annotated) denote formulae. Upper case letters from the end of the alphabet like X, Y, Z (possibly annotated) denote *finite* (possibly empty) sets of formulae.

The logics we consider are all normal extensions of the minimal normal modal logic **K** and are axiomatised by taking the rules of necessitation and modus ponens as inference rules, by taking the axiom schemas of classical propositional logic, and by taking K and combinations of formulae from Figure 2.1 as further axiom schemas. For an introduction to modal logics see [HC84].

The name of a logic is usually formed by concatenating the names of its (modal) axiom schemas to **K** to denote that the logic is a normal extension of **K**. However we use the traditional names of the more famous logics; for example **S4** is **KT4** and **S4.2** is **KT42**. The logic **S4R** is also known in the literature as **S4.4** and as **SW5**, while **S4F** is also known as **S4.3.2** [Seg71].

2.2 Deducibility and Theoremhood

A deduction of a formula A in logic **L** from a finite set of formulae Γ is a finite sequence of formulae $A_1, A_2, \cdots, A_n$ such that $A_n = A$ and each A_i is: (1) a member of Γ; or (2) an instance of an axiom schema of **L**; or (3) equal to $\Box A_j$ for some $j < i$; or (4) obtained from some A_j and A_k via modus ponens where $k < i$ and $j < i$. We write $\Gamma \vdash_L A$ to indicate there is a deduction of A in **L** from Γ.

This notion of deduction is stronger than the one usually employed in modal logic where "$\Gamma \vdash_L A$" corresponds to the statement that for some finite subset $\{A_1, A_2, \cdots, A_n\}$ of Γ, we have $\vdash_L A_1 \wedge A_2 \wedge \cdots \wedge A_n \Rightarrow A$ [Gol87]. For example, $p \vdash_L \Box p$ is a perfectly legal deduction in our formulation but $\vdash_L p \Rightarrow \Box p$ is rarely a theorem in modal logics. If Γ is empty then the two

Axiom Name	Defining Formula	Alternative Names
K	$\Box(A \Rightarrow B) \Rightarrow (\Box A \Rightarrow \Box B)$	
T	$\Box A \Rightarrow A$	M
4	$\Box A \Rightarrow \Box\Box A$	
5	$\Diamond A \Rightarrow \Box\Diamond A$	E
D	$\Box A \Rightarrow \Diamond A$	
R	$\Diamond\Box A \Rightarrow (A \Rightarrow \Box A)$	W5
F	$(\Box A \Rightarrow B) \vee (\Diamond\Box B \Rightarrow A)$	
2	$\Diamond\Box A \Rightarrow \Box\Diamond A$	

Figure 1: Axiom names and associated schemas.

notions coincide and we say that A is a **theorem** of **L** if $\vdash_L A$. Clearly, the deduction theorem is the key but for most of this paper we shall deal with the weaker notion called theoremhood.

2.3 Kripke Semantics

We assume familiarity with the notions of Kripke frames $\langle W, R\rangle$ and Kripke models $\langle W, R, V\rangle$. A possible world w satisfies an atomic formula p if and only if $w \in V(p)$. We write this as $w \models p$ and write $w \not\models p$ to mean "not $w \models p$". The semantics of the other connectives and modal operators are as usual.

We often use annotated names like w_1 and w_2 to denote possible worlds. Unless stated explicitly, there is no reason why w_1 and w_2 cannot name the same world. We use R as a name of an axiom schema and also for the reachability relation but the context should make clear which R is meant.

In any model $\langle W, R, V\rangle$, a formula A is **true in a world** $w \in W$ if $w \models A$. A formula A is **valid in a model** $\mathcal{M} = \langle W, R, V\rangle$, written as $\mathcal{M} \models A$, if it is true in all worlds in that model. A formula A is **valid in a frame** $\mathcal{F} = \langle W, R\rangle$, written as $\mathcal{F} \models A$, if A is valid in all models based on $\mathcal{F}$. Suppose $\mathcal{C}$ is a class of models, or of frames. A formula A is **valid in a class** $\mathcal{C}$, written as $\mathcal{C} \models A$, if it is valid in every member of $\mathcal{C}$. An axiom is said to be valid in a model (valid in a frame) if all instances of that axiom have that property. If we have a set of formulae X then $\mathcal{M} \models X$ ($\mathcal{F} \models X$) denotes that all members of X are valid in $\mathcal{M}$ ($\mathcal{F}$).

Let $\mathcal{C}$ be either a collection of models, or of frames. Then logic **L** is **sound with respect to** $\mathcal{C}$ if for every formula A we have that $\vdash_{\mathbf{L}} A$ implies $\mathcal{C} \models A$ [HC84]. Logic **L** is **complete with respect to** $\mathcal{C}$ if for every formula A we have that $\mathcal{C} \models A$ implies $\vdash_{\mathbf{L}} A$ [HC84]. A logic **L** is **determined** or **characterised** by a class $\mathcal{C}$ if it is both sound and complete with respect to $\mathcal{C}$; that is, when $\mathcal{C} \models A$ iff $\vdash_{\mathbf{L}} A$.

A frame $\langle W, R\rangle$ is: **reflexive** if $\forall w \in W, wRw$; **transitive** if $\forall w_1, w_2, w_3 \in W, w_1Rw_2$ and w_2Rw_3 implies w_1Rw_3; **serial** if $\forall w_1 \in W, \exists w_2 \in W, w_1Rw_2$; **symmetric** if $\forall w_1, w_2 \in W, w_1Rw_2$ implies w_2Rw_1; and **convergent** if $\forall w_1, w_2 \in W, \exists w_3 \in W, w_1Rw_3$ and w_2Rw_3 (where i and j are positive integers).

If $\langle W, R\rangle$ is a frame where R is transitive, then a **cluster** C is a maximal subset of W such that for all *distinct* worlds w and w' in C we have wRw' and $w'Rw$. A cluster is **degenerate** if it is a single irreflexive world, otherwise it is **nondegenerate**. A nondegenerate cluster is **proper** if it consists of two or more worlds. A nondegenerate cluster is **simple** if it consists of a single reflexive world. Note that in a nondegenerate cluster, R is reflexive, transitive *and symmetric*. If the frame is transitive and convergent then there is guaranteed to be a **last** cluster which is reachable from every other cluster. In the case where the clusters form a tree (or a linear sequence), the leaf nodes are known as **final clusters**. For an introduction to Kripke frames, Kripke models and the notion of clusters see Hughes and Cresswell [HC84].

A frame is an **L**-frame if it meets the conditions for **L** as shown in Figure 2. It is known that logic **L** is characterised by the class of all **L**-frames [Seg71, pages 77-78 and 160], [Sch]. Then, a model $\mathcal{M} = \langle W, R, V\rangle$ is an **L-model** if $\langle W, R\rangle$ is an **L**-frame. Also, $\mathcal{M}$ is an **L-model for** (a finite set of formulae) X if there exists $w \in W$ such that $w \models X$. Recall that $w \models X$ means that $w \models A$ for all $A \in X$. A formula A is **L-valid** iff A is valid in all **L**-models, and hence in all **L**-frames. A finite set X is **L-satisfiable** iff there exists an **L**-model for X. So, X is **L-unsatisfiable** iff there are no **L**-models for X.

L	**L**-frame
K45	type 1: a lone cluster (degenerate or nondegenerate); or type 2: a degenerate cluster followed by a nondegenerate cluster
K45D	type 1: a lone nondegenerate cluster; or type 2: a degenerate cluster followed by a nondegenerate cluster
S4R	type 1: a lone nondegenerate cluster; or type 2: a simple cluster followed by a nondegenerate cluster
S4F	a sequence of at most two nondegenerate clusters
S4	any reflexive and transitive frame
S4.2	any reflexive, transitive and convergent frame

Figure 2: Definition of **L**-frames.

3 Semantic Tableaux and Tableau Rules

Since our tableau systems work with *finite* sets of formulae, we use the following notational conventions: (1) P and Q stand for formulae; (2) U, X, Y, Z stand for finite (possibly empty) sets of formulae; (3) "$X; Y$" stands for $X \cup Y$; (4) "$X; P$" stands for $X \cup \{P\}$; (5) $\Box X$ stands for $\{\Box P \mid P \in X\}$ and (6) $\neg\Box X$ stands for $\{\neg\Box P \mid P \in X\}$. To minimise the number of rules, we work with primitive notation in terms of $\neg, \Box$ and $\wedge$. Each of our tableau rules has a dual rule which can be easily obtained by using the definition of $\Diamond$ as $\neg\Box\neg$. The tableau systems and the completeness proofs are based on those of Rautenberg [Rau83].

3.1 Syntax of Tableau Systems

Tableau systems consist of a collection of tableau (inference) rules. A tableau rule consists of a **numerator** above the line and a list of **denominators** (below the line). The denominators are separated by vertical bars. The numerator is a finite set of formulae and so is each denominator. We use the terms numerator and denominator rather than premiss and conclusion to avoid confusion with the sequent terminology.

Figure 3 shows the tableau rules we require. Each tableau rule is read downwards as "if the numerator is **L**-satisfiable, then so is one of the denominators". A tableau calculus is a finite collection of tableau rules identified with the set of its rule names. Thus the tableau calculus for propositional classical logic **PC** is $\mathcal{C}PC = \{(0), (\neg), (\theta), (\wedge), (\vee)\}$. The other tableau calculi we consider are shown in Figure 4. The $(S4.2)$ rule is the only potentially dangerous rule since its denominator contains a formula to which the rule can be applied in an endless fashion. To forbid this the new formula is marked with a star and the $(S4.2)$ rule is restricted to apply only to non-starred formulae. All other rules must treat starred formula as if they were non-starred.

Following Rautenberg [Rau83, Rau85], a $\mathcal{C}L$**-tableau** for X is a finite tree $\mathcal{T}$ with root X whose nodes carry finite formula sets stepwise constructed by the rules of $\mathcal{C}L$ according to: if a rule with n denominators is applied to a node then that node has n successors with the proviso that if a node E carries a set Y and Y has already appeared on the branch from the root to E then E is an end node of $\mathcal{T}$. A tableau is **closed** if all its end nodes carry $\{0\}$. A set X is $\mathcal{C}L$**-consistent** if no closed $\mathcal{C}L$-tableau for X exists.

When formulated using sets rather than multisets, tableau systems include an implicit rule of contraction since the set $X; P; P$ is the same as the set $X; P$. In order to eliminate contraction, we explicitly build contraction into the rules. For example, the (T) rule contains a form of

$$(\wedge)\ \frac{X; P \wedge Q}{X; P; Q} \qquad (0)\ \frac{P; \neg P}{0} \qquad (\vee)\ \frac{X; \neg(P \wedge Q)}{X; \neg P \ \mid\ X; \neg Q}$$

$$(\neg)\ \frac{X; \neg\neg P}{X; P} \qquad (\theta)\ \frac{X; Y}{X} \qquad (T)\ \frac{X; \Box P}{X; \Box P; P}$$

$$(45)\ \frac{\Box X; \neg\Box Y; \neg\Box P}{X; \Box X; \neg\Box Y; \neg\Box P; \neg P} \qquad (S4)\ \frac{\Box X; \neg\Box P}{\Box X; \neg P}$$

$$(45D)\ \frac{\Box X; \neg\Box Y; \neg\Box P}{X; \Box X; \neg\Box Y; \neg\Box P; \neg P} \quad \text{where } Y; P \text{ may be empty}$$

$$(R)\ \frac{X; \neg\Box P}{X; \neg\Box P; \neg P \ \mid\ X; \neg\Box P; \Box\neg\Box P; P}$$

$$(S4F)\ \frac{U; \Box X; \neg\Box P; \neg\Box Y}{U; \Box X; \neg\Box P; \neg\Box Y; \Box\neg\Box P \ \mid\ \Box X; \neg\Box P; \neg\Box Y; \neg P}$$

$$(S4.2)\ \frac{X; \neg\Box P}{X; \neg\Box P; \Box\neg\Box P \ \mid\ X; \neg\Box P; \Box(\neg\Box\neg\Box P)^*} \quad \text{where } \neg\Box P \text{ is not starred}$$

$$(sfc)\quad \frac{X; \neg(P \wedge Q)}{X; \neg P; \neg Q \mid X; \neg P; Q \ \mid\ X; P; \neg Q} \qquad \frac{X; \neg\Box P}{X; \neg\Box P; P \mid X; \neg\Box P; \neg P} \qquad \frac{X; \Box P}{X; \Box P; P \mid X; \Box P; \neg P}$$

$$(sfcT)\quad \frac{X; \neg(P \wedge Q)}{X; \neg P; \neg Q \mid X; \neg P; Q \ \mid\ X; P; \neg Q} \qquad \frac{X; \neg\Box P}{X; \neg\Box P; P \mid X; \neg\Box P; \neg P}$$

Figure 3: Tableau rules.

$\mathcal{C}L$	Static Rules	Transitional Rules	Structural Rules
$\mathcal{C}PC$	$(0), (\neg), (\wedge), (\vee)$	–	(θ)
$\mathcal{C}S4$	$(0), (\neg), (\wedge), (\vee), (T)$	$(S4)$	(θ)
$\mathcal{C}K45$	$(0), (\neg), (\wedge), (\vee)$	(45)	(θ)
$\mathcal{C}K45D$	$(0), (\neg), (\wedge), (\vee)$	$(45D)$	(θ)
$\mathcal{C}S4R$	$(0), (\neg), (\wedge), (\vee), (T), (R)$	$(S4)$	(θ)
$\mathcal{C}K45\dagger$	$(0), (\neg), (\wedge), (sfc)$	(45)	(θ)
$\mathcal{C}K45D\dagger$	$(0), (\neg), (\wedge), (sfc)$	$(45D)$	(θ)
$\mathcal{C}S4.2\dagger$	$(0), (\neg), (\wedge), (sfcT), (T), (S4.2)$	$(S4)$	(θ)
$\mathcal{C}S4F\dagger$	$(0), (\neg), (\wedge), (sfcT), (T), (S4.2)$	$(S4F), (S4)$	(θ)

Figure 4: Tableau Calculi

contraction on $\Box P$ since $\Box P$ is carried from the numerator into the denominator. We return to this point later in Section 6.

For a formula A, the *finite* set of all subformulae $Sf(A)$ is defined inductively as usual where $A \in Sf(A)$. For any finite set X :

- let $Sf(X)$ denote the set of all subformulae of all formulae in X;
- let $\neg Sf(X)$ denote $\{\neg P \mid P \in Sf(X)\}$;
- let $\widetilde{X}$ denote the set $Sf(X) \cup \neg Sf(X) \cup \{0\}$;
- let $X^*_{\mathcal{C}K45} = X^*_{\mathcal{C}K45\dagger} = X^*_{\mathcal{C}K45D} = X^*_{\mathcal{C}45D\dagger} = \widetilde{X}$;
- let $X^*_{\mathcal{C}S4R} = Sf(\Box\widetilde{X})$.
- let $X^*_{\mathcal{C}S4.2\dagger} = X^*_{\mathcal{C}S4F\dagger} = Sf(\Box\neg\Box\widetilde{X})$.

Thus, the set carried by *any* node in *any* $\mathcal{C}L$-tableau for X must be a subset of $X^*_{\mathcal{C}L}$. A tableau system $\mathcal{C}L$ has the subformula property if $X^*_{\mathcal{C}L} = \widetilde{X}$, and when this holds, $\mathcal{C}L$ is said to be **analytic** [Fit83]. If $\widetilde{X} \subset X^*_{\mathcal{C}L}$ then $\mathcal{C}L$ breaks the subformula property and is said to be **semi-analytic** since $\mathcal{C}L$ may not be a decision procedure. However, as long as $X^*_{\mathcal{C}L}$ is *finite* then any $\mathcal{C}L$-tableau for X is guaranteed to terminate (for finite X) and there are only a finite number of such tableaux, so that even a semi-analytic $\mathcal{C}L$ *is* a decision procedure. Some of our systems are semi-analytic since they break the subformula property, and some even contain an (analytic) cut rule, but in each case, $X^*_{\mathcal{C}L}$ is finite.

Intuitively, we can associate the numerator and the denominator of a tableau rule with possible worlds. We classify a rule as a **static rule** if the numerator and the denominator correspond to the same world and classify a rule as a **transitional rule** if at least one of the denominators corresponds to a different world. Note that there is only one explicit structural rule (θ).

A set X is **closed with respect to a tableau rule** if, whenever (an instantiation of) the numerator of the rule is in X, so is (a corresponding instantiation of) at least one of the denominators of the rule. If $\mathcal{C}$ is a finite collection of tableau rules then a set X is **closed with respect to** $\mathcal{C}$ if it is closed with respect to each rule in $\mathcal{C}$. For each $\mathbf{L}$, a set X is $\mathcal{C}L$**-saturated** if it is $\mathcal{C}L$-consistent and closed with respect to the static rules of $\mathcal{C}L$.

Lemma 1 *For each $\mathcal{C}L$-consistent X there is an effective procedure to construct some finite $\mathcal{C}L$-saturated X^* with $X \subseteq X^* \subseteq X^*_{\mathcal{C}L}$.*

Proof: The proof is based on the facts that: (0) is not applicable at any stage since no $\mathcal{C}L$-tableau for X closes by supposition; that each sequence of rule applications is guaranteed to terminate (possibly with a cycle); and that each rule application is guaranteed to give at least one $\mathcal{C}L$-consistent denominator. That is, X^* is obtained as the union of stepwise tableau rule applications. Since each tableau rule maps a subset of $X^*_{\mathcal{C}L}$ to another subset of $X^*_{\mathcal{C}L}$, we must have $X^* \subseteq X^*_{\mathcal{C}L}$. •

Such $\mathcal{C}L$-saturated sets are important because they provide a direct connection between the syntactic and semantic aspects of tableau systems. This is the subject of the next section.

3.2 Soundness and Completeness of Modal Tableau Systems

For soundness we have to show that for each $\mathcal{C}$L-tableau rule: if the numerator is **L**-satisfiable then at least one of the denominators is **L**-satisfiable. For completeness we have to show that if there is no closed $\mathcal{C}L$-tableau for X then X has an **L**-model (i.e. there is an **L**-model which is an **L**-model for X). The basic idea is due to Hintikka.

The following definition from Rautenberg [Rau83] is central for the model construction mentioned above. A **model graph** for some finite fixed set of formulae X is a finite **L**-frame $\langle W, R\rangle$ such that all $w \in W$ are $\mathcal{C}L$-saturated sets with $w \subseteq X^*_{\mathcal{C}L}$ and

(i) $X \subseteq w_0$ for some $w_0 \in W$;

(ii) if $\neg\Box P \in w$ then there exists some $w' \in W$ with wRw' and $\neg P \in w'$;

(iii) if wRw' and $\Box P \in w$ then $P \in w'$.

Lemma 2 *If $\langle W, R\rangle$ is a model graph for X then there exists an* **L***-model for X at node w_0 [Rau83].*

This model graph construction is similar in spirit to the subordinate frames construction of Hughes and Cresswell [HC84] except that they use maximal consistent sets and do not consider cycles, giving infinite models rather than finite models.

4 Soundness and Completeness

4.1 Soundness

Theorem 1 (soundness) *If* **L** *is one of* **K45**, **K45D**, **S4**, **S4R**, **S4.2** *and* **S4F** *then the tableau calculi $\mathcal{C}L$ and $\mathcal{C}L\dagger$ are sound with respect to* **L***-frames.*

Proof : It is easy to show that the rules (0), ($\neg$), ($\wedge$), (θ), and ($\vee$) are sound with respect to all our **L**-frames. The ($sfcT$) rule is merely a reflexive counterpart of the (sfc) rule. The soundness proofs of the modal rules are very similar so we give the intuitions behind the more arcane rules.

Intuitions for the ($S4$) **rule:** If $w \models \neg\Box P$ then $w \models \Diamond\neg P$. Hence there is some w' with wRw' such that $w' \models \neg P$. The $\Box$-formulae of w can be carried into w' by transitivity. Hence the numerator of $(S4)$ represents w and the numerator represents w' and $(S4)$ is a transitional rule.

Intuitions for the $(S4.2)$ **rule:** Suppose $w \models \neg\Box P$. Consider some w' in the last cluster. By the law of excluded middle $w' \models \Box\neg\Box P$ or $w' \models \neg\Box\neg\Box P$. If $w' \models \Box\neg\Box P$ then $w \models \Box\neg\Box P$. Else, if $w' \models \neg\Box\neg\Box P$ then $w \models \Box\neg\Box\neg\Box P$. Since the numerator and both denominators represent the same world $(S4.2)$ is a static rule.

Intuitions for the $(S4F)$ **rule:** Assume the model has exactly two clusters C_1 and C_2. If $w \models \neg\Box P; \neg\Box Y$ then there is some w' such that wRw' and $w' \models \neg P$. If w' occurs in C_1 then w must also occur in C_1, hence $w' \models \neg\Box P; \neg\Box Y$. Else if w' occurs in C_2 then all worlds in the model must satisfy $\Box\neg\Box P$. In particular, $w \models \Box\neg\Box P$. The right denominator represents the transition to w' whereas the left denominator represents w.

Note that, in general, the denominators of the transitional rules do not inherit *all* the formulae of the numerator. Thus, the transitional rules involve a loss of semantic information about the contents of the parent node.

4.2 Completeness of the cut-free systems

Theorem 2 (completeness) *If X is a finite set of formulae and X is $\mathcal{C}L$-consistent then there is an $\mathbf{L}$-model for X on a finite $\mathbf{L}$-frame where $\mathbf{L} \in \{\mathbf{S4}, \mathbf{K45}, \mathbf{K45D}, \mathbf{S4R}\}$. See [Gor93].*

4.3 Completeness for Tableau Systems Containing Analytic Cut

The tableau systems for **S4F** and **S4.2** require the analytic cut rule (sfc) of Smullyan for completeness. Adding analytic cut to the systems for **K45** and **K45D** simplifies the completeness proofs and gives a more direct satisfiability test, as shown below.

A set X is **subformula-complete** if $P \in Sf(X)$ implies that either $P \in X$ or $\neg P \in X$. A crucial consequence of adding analytic cut is the following lemma.

Lemma 3 *If X is closed with respect to $\{(0), (\neg), (\wedge), (sfc)\}$ or with respect to $\{(0), (\neg), (\wedge), (sfcT), (T)\}$ then X is subformula-complete.*

Also note that if $w \prec v$ then $v \subseteq \widetilde{w}$ where $\widetilde{w} = w \cup \neg w \cup \{0\}$; that is, the analytic cut rule can introduce new formulae of the form $\neg P$ into v only if $P \in Sf(w)$. This is crucial for the completeness proofs.

Theorem 3 (completeness) *If X is a finite set of formulae and X is $\mathcal{C}L\dagger$-consistent then there is an $\mathbf{L}$-model for X on a finite $\mathbf{L}$-frame where $\mathbf{L} \in \{\mathbf{K45}, \mathbf{K45D}, \mathbf{S4.2}, \mathbf{S4F}\}$. See [Gor93].*

5 Decision Procedures

Each $\mathcal{C}L$ gives a nondeterministic procedure to decide whether or not some given (finite) formula A is a theorem of logic $\mathbf{L}$. To test whether a formula A is $\mathbf{L}$-valid, we simply have to run a $\mathcal{C}L$-tableau construction for $X = \{\neg A\}$. Since $X^*_{\mathcal{C}L}$ is finite, there are only a finite number of such $\mathcal{C}L$-tableaux. If one of them is closed then, by soundness, $\{\neg A\}$ has no $\mathbf{L}$-models, and hence A is $\mathbf{L}$-valid. If none of these $\mathcal{C}L$-tableaux closes then, by completeness, we can construct a finite **L**-model $\langle W, R, \vartheta \rangle$ which satisfies $\neg A$, hence A is not **L**-valid. We already know that the axiomatically formulated logic **L** is characterised by **L**-frames. That is, we know that a formula A is **L**-valid iff $\vdash_L A$. Therefore, each $\mathcal{C}L$ is a highly nondeterministic decision procedure for theoremhood in **L**.

Furthermore, since each completeness proof is constructive, each proof gives a *deterministic* procedure to test whether an arbitrary (finite) set of formulae X is **L**-satisfiable. However, it is known that the satisfiability problem is PSPACE-complete for propositional **S4** and NP-complete for propositional **S5** [HM85]. A similar argument shows that the satisfiability problem for **K45**, **K45D**, **S4R** and **S4F** are also NP-complete whilst the satisfiability problem for **S4.2** is PSPACE-complete.

6 Eliminating Thinning and Contraction

In all of our modal tableau systems, the only explicit structural rule is the rule of thinning (θ). The thinning (or weakening) rule introduces a form of nondeterminism where we have to guess which formula to throw away. That is, we have to guess which formulae are really essential to the proof.

The (θ) rule can be eliminated from all our systems by building the effects of (θ) into the transitional rules and simultaneously changing the basic axiomatic tableau rule from (0) to (0′) as shown below

$$(0') \quad \frac{X; P; \neg P}{0}$$

This technique is used by Fitting [Fit83] and also works for all the other modal calculi we have presented. See [Gor93].

Some of the tableau rules we have given are not standard; for example, the (T) rule is usually given as:

$$\frac{X; \Box P}{X; P} \ (T')$$

where $\Box P$ is not carried from the numerator into the denominator [Rau83]. It is well known that the rule of contraction, which is implicit in the set formulation, then becomes *essential* for completeness. It is also well known that although contraction becomes essential, it is required *only* for $\Box$-formulae in most normal modal logics, and on both $\Box$-formulae and $\Diamond$-formulae in some symmetric normal modal logics [Fit88]. We have deliberately built contraction into our rules to highlight this fact. We believe that if we interpret ";" as multiset union, and rework our formulation using multisets instead of sets, then all the proofs will still go through with appropriate modifications. That is, the rule of contraction appears to be *eliminable* from our systems as long as the static rules build in contraction as given by our rules. Unfortunately, the proofs become very messy.

Therefore these systems are amenable to the Prolog implementation technique of Fitting [Fit88].

7 Deducibility and Strong Completeness

The notion of deducibility, $\Gamma \vdash_L A$, which we use is stronger than the usual one in modal logic where the weaker notion of theoremhood, $\vdash_L A$, is standard. The semantic notion of characterisation by a class of frames must be strengthened to mirror this change. A modal logic **L** is **strongly characterised** by a class of frames $\mathcal{C}$ if for every set of formulae Γ and every formula A,

$$\Gamma \vdash_L A \quad \text{iff} \quad \forall V,\ \text{if } \langle W, R\rangle \in \mathcal{C} \text{ and } \langle W, R, V\rangle \models \Gamma \text{ then } \langle W, R, V\rangle \models A.$$

It is known that the weak characterisation results of modal logics can be strengthened to strong characterisation results if the modal logic $\mathbf{L}$ is canonical and the class of frames $\mathcal{C}$ contains the canonical frame for $\mathbf{L}$ [McD82, MST91]; see Hughes and Cresswell [HC84] for the meanings of canonicity and canonical frame. The logics we study are known to be strongly characterised by the class of frames shown in Figure 2 because these logics and the corresponding $\mathbf{L}$-frames satisfy these two conditions.

We can extend our tableau systems to decide whether Γ deduces A, that is, if $\Gamma \vdash_L A$. The trick is to start a tableau for $\Gamma \cup \{\neg A\}$ and then to simply add all members of Γ to each new tableau node as it is created. The set Γ now has "global" status, the proofs of soundness and completeness still go through, and the completeness proofs remain constructive. We are now searching for a countermodel for A where every world satisfies Γ. If we find a closed tableau then we know that no such countermodel is possible and hence, by soundness, that any $\mathbf{L}$-model validating Γ must validate A; that is, $\Gamma \vdash_L A$. If we find no closed tableau then, by completeness, we can construct an $\mathbf{L}$-model validating Γ and containing a world w_0 which falsifies A. Hence $\Gamma \not\vdash_L A$.

An easier way to achieve the same effect is to utilise the fact that all of our transitional rules preserve $\Box$-formulae. That is, for $\mathbf{L} \in \{\mathbf{K45}, \mathbf{K45D}\}$ we run a normal tableau for $\Gamma; \Box\Gamma; \neg A$ to decide whether $\Gamma \vdash_L A$. For $\mathbf{L} \in \{\mathbf{S4}, \mathbf{S4R}, \mathbf{S4.2}, \mathbf{S4F}\}$ we run a normal tableau for $\Box\Gamma; \neg A$. In each case, the members of Γ will be added to each node automatically by $\mathcal{C}L$.

Let $\widehat{\Gamma}$ denote the conjunction of all formulae in Γ. Then, the fact that our tableau rules extend to handle (strong) deducibility indicates that a form of the deduction theorem goes through because: $\Gamma \vdash_L A$ iff $\vdash_L (\widehat{\Box\Gamma} \wedge \widehat{\Gamma}) \Rightarrow A$ for $\mathbf{L} \in \{\mathbf{K45}, \mathbf{K45D}\}$ and $\Gamma \vdash_L A$ iff $\vdash_L \widehat{\Box\Gamma} \Rightarrow A$ for $\mathbf{L} \in \{\mathbf{S4}, \mathbf{S4R}, \mathbf{S4.2}, \mathbf{S4F}\}$.

A caveat is in order because our tableaux procedures terminate only because we deal with *finite* sets. Consequently, we cannot handle first order nonmonotonic modal theories where the infinite set Γ contains all ground instances of the formulae in the (real) first order theory Γ'.

8 Sequent Systems

For each tableau rule there is an analogous sequent rule associating the tableau set X with the sequent $\Sigma \longrightarrow \Delta$ by putting $X = (\Sigma \cup \neg\Delta)$ and turning the tableau rule upside down. For example, the sequent analogue of the tableau rule (45) is

$$\frac{\Box\Gamma \longrightarrow \Box\Delta, \Box A, A}{\Box\Gamma \longrightarrow \Box\Delta, \Box A}(\rightarrow \Box : K45)$$

Thus, each tableau system $\mathcal{C}L$ has an analogous sequent system $\mathcal{S}L$.

These sequent systems then give a Gentzen characterisation of theoremhood $\vdash_L$ via the equivalence: $\vdash_L A$ iff the sequent $\longrightarrow A$ is provable in $\mathcal{S}L$. In order to handle (strong) deducibility Fitting [Fit83] introduces the notation $\Gamma \Vdash \Sigma \longrightarrow A$ where $\Sigma \longrightarrow A$ takes the role mentioned above and $\Gamma \Vdash \Sigma \longrightarrow A$ takes the role of $\Gamma \vdash_L \widehat{\Sigma} \Rightarrow A$. Note that the sequent arrow changes to classical implication in this transformation and $\widehat{\Sigma}$ denotes the conjunction of all formulae in Σ. We can obtain $\Gamma \vdash_L A$ by putting Σ to be empty.

Some sort of distinction must be made between Γ and Σ because the former are like "global assumptions" while the latter are like "local assumptions" [Fit83]. If we work with transitive logics then we can use the same trick as in our tableau method by asking whether the sequent $\Gamma, \Box\Gamma, \Sigma \longrightarrow A$ is provable, allowing us to drop the distinction between Γ and Σ. This trick will not work for non-transitive logics because the tableau/sequent rules no longer remain sound with the intended semantics.

9 Related Work

The cut-free tableau systems $\mathcal{C}K45$ and $\mathcal{C}K45D$ are the tableau versions of Schwarz's sequent systems for **K45** and **K45D** [Shv89]. The systems $\mathcal{C}K45\dagger$ and $\mathcal{C}K45D\dagger$ are based on the work of Rautenberg [Rau83]. The advantage of $\mathcal{C}K45\dagger$ and $\mathcal{C}K45\dagger$ is that the associated deterministic satisfiability test is purely local. That is, we need only a two level graph representation, and there is no need to check for repeated nodes.

Apparently Serebriannikov has also obtained a sequent system for **S4.2** but I have been unable to trace this work, let alone determine if this system is cut-free. I know of no other systems for **S4F** or **S4.2** although the idea of the $(S4.2)$ rule is due to Rautenberg [Rau83].

Zeman [Zem73] gives a tableau system for **S4R** which he calls **S4.4** but Zeman's system is not cut-free, requiring analytic cut. Zeman also gives a sequent system for **S4R** which is an amalgamation of his sequent systems for **S4** and **S5**. Note however that Zeman explicitly uses an **S5** system and his system for **S5** is not cut-free whereas we simply add one extra static rule to $\mathcal{C}S4$ to obtain $\mathcal{C}S4R$.

Fitting [Fit83] gives strongly complete analytic tableau systems for many modal logics including **S4** but does not give (strongly complete) systems for the logics we have considered. Incidentally, Rautenberg's technique also gives a strong *analytic* tableau system (containing an analytic cut rule) for **S5**.

Catach [Cat88] has implemented an automatic theorem prover in Prolog for many modal logics. His method is essentially the "method of semantic diagrams" of Hughes and Cresswell [HC68] where the reachability relation is represented explicitly and the various constraints like reflexivity and transitivity are also enforced explicitly. We have *deliberately* avoided this method because it does not yield a proof theory for the logic in question. However, such "explicit" methods can be used to implement the deterministic decision procedures obtained from our completeness proofs.

10 Further Work

Segerberg [Seg71] shows that **S4F** can also be axiomatised as **S4.3** plus the axiom schema $\Diamond(\Box P \wedge \Diamond\Box Q \wedge \neg Q) \Rightarrow P$ where **S4.3** itself is **S4** plus the axiom schema $\Box(\Box P \Rightarrow Q) \vee \Box(\Box Q \Rightarrow P)$. Elsewhere we have given cut-free tableau systems for **S4.3** and other Diodorean modal logics [Gor92]. We believe that a much more elegant system for **S4F** is possible based on the system for **S4.3**.

Our counter-model constructions for $\mathcal{C}S4R$ and $\mathcal{C}S4F\dagger$ are parasitic on the basic $\mathcal{C}S4$ counter-model construction because, in both cases, we have to wait until "eventually this process cycles". Are there more direct counter-model constructions for $\mathcal{C}S4F\dagger$ like the ones for $\mathcal{C}K45\dagger$? Does adding analytic cut to $\mathcal{C}S4R$ help ?

11 Conclusions

We have given (semi)analytic tableau systems for the propositional (monotonic) modal logics **K45**, **K45D**, **S4R**, **S4F** and **S4.2**. Each system provides a nondeterministic decision procedure for theoremhood $\vdash_L A$ and each completeness proof gives a deterministic **L**-satisfiability test. Each tableau system has a sequent analogue giving a Gentzen cut-elimination theorem for **S4R** from the fact that $\mathcal{C}S4R$ is cut-free. These systems extend trivially to handle (strong) deducibility, $\Gamma \vdash_L A$, as long as (strong) deducibility allows necessitation on members of the theory Γ. This is usually forbidden in modal logic but is essential for nonmonotonic modal logics.

Recent results indicate that the *nonmonotonic* modal logics based upon **K45**, **K45D**, **S4R**, **S4F** and **S4.2** elegantly generalise various *nonmodal* nonmonotonic formalisms like autoepistemic logics and default logics. Indeed, the currently known algorithms for computing the nonmonotonic consequences of a theory Γ depend on decision procedures for (strong) deducibility in the underlying monotonic modal logic. Therefore our systems are important for automating nonmonotonic deduction in both modal and nonmodal formulations. The limitation is that Γ must be finite, but when this holds, our tableau calculi are directly implementable in Prolog.

Acknowledgements: I am grateful to Harold Simmons for many helpful discussions about strong deducibility, and modal logic in general.

References

[Cat88] L Catach. TABLEAUX: A general theorem prover for modal logics. In *Proc. International Computer Science Conference: Artificial Intelligence: Theory and Applications*, pages 249–256, 1988.

[Fit83] M. Fitting. *Proof Methods for Modal and Intuitionistic Logics*, volume 169 of *Synthese Library*. D. Reidel, Dordrecht, Holland, 1983.

[Fit88] M. Fitting. First order modal tableaux. *Journal of Automated Reasoning*, 4:191–213, 1988.

[Gol87] R. I. Goldblatt. *Logics of Time and Computation*. CSLI Lecture Notes Number 7, CSLI Stanford, 1987.

[Gor92] Rajeev Goré. The cut-elimination theorem for Diodorean modal logics. Technical Report Forthcoming, University of Cambridge, England, 1992. Abstract in the Proc. of the Logic Colloquium, Veszprem, Hungary, 1992.

[Gor93] Rajeev Goré. Semi-analytic tableaux for modal logics of nonmonotonicity. Technical Report forthcoming, University of Manchester, England, 1993.

[HC68] G. E. Hughes and M. J. Cresswell. *Introduction to Modal Logic*. Methuen, London, 1968.

[HC84] G. E. Hughes and M. J. Cresswell. *A Companion to Modal Logic*. Methuen, London, 1984.

[HM85] J. Y. Halpern and Y. Moses. A guide to the modal logics of knowledge and belief: Preliminary draft. In *Proc. IJCAI*, pages 480–490, 1985.

[McD82] D. McDermott. Nonmonotonic logic II: Nonmonotonic modal theories. *JACM*, 29:33–57, 1982.

[MST91] W. Marek, G. Schwarz, and M. Truszczynski. Modal nonmonotonic logics: ranges, characterisation, computation. Technical Report 187-91, Dept. of Computer Science, University of Kentucky, USA, 1991.

[Rau83] W. Rautenberg. Modal tableau calculi and interpolation. *Journal of Philosophical Logic*, 12:403–423, 1983.

[Rau85] W. Rautenberg. Corrections for modal tableau calculi and interpolation by W. Rautenberg, JPL 12 (1983). *Journal of Philosophical Logic*, 14:229, 1985.

[Sch] Grigory Schwarz. Minimal model semantics for nonmonotonic modal logics. In *Proc. LICS-92*.

[Seg71] Krister Segerberg. An essay in classical modal logic (3 vols.). Technical Report Filosofiska Studier, nr 13, Uppsala Universitet, Uppsala, 1971.

[Shv89] Grigori F. Shvarts. Gentzen style systems for K45 and K45D. In A. R. Meyer and M. A. Taitslin, editors, *Logic at Botik '89, Symposium on Logical Foundations of Computer Science, LNCS 363*, pages 245–256. Springer-Verlag, 1989.

[ST] G. E. Schwarz and M. Truszczynski. Modal logic S4F and the minimal knowledge paradigm. In *Proc. of TARK 1992*.

[Trua] M. Truszczynski. Embedding default logic into modal nonmonotonic logics. In W. Marek, A. Nerode, and V. S. Subramanian, editors, *Proc. of the first international workshop on logic programming and non-monotonic reasoning*.

[Trub] M. Truszczynski. Modal interpretations of default logic. In *IJCAI-91*.

[Zem73] J. J. Zeman. *Modal Logic: The Lewis-Modal Systems*. Oxford University Press, 1973.