Computer Science
University of Manchester

The GLUEING CONSTRUCTION and LAX LIMITS (with applications to categories of structured posets)

Harold Simmons

Technical Report UMCS–93–6–2

THE GLUEING CONSTRUCTION
and
LAX LIMITS
(with applications to categories of structured posets)

Harold Simmons

Department of Computer Science
The University
Manchester M13 9PL
`hsimmons@cs.man.ac.uk`

June 14, 1993

Abstract

Starting life as a way of reconstructing a topological space from a pair of complementary subspaces, the glueing construction has found employment in a wide range of different roles from the construction of free distributive lattices to a supporting part in the 2-categorical analysis of types theories. In this latter role the construction appears to be a fundamental factor in the behaviour of higher order proof theory. What is going on here? Before that can be answered we need at least a less ad hoc description of the construction.

In this paper I set down what is, I believe, the beginnings of a coherent account of the algebraic version of glueing. As well as the abstract theory I give a good selection of different examples to illustrate the diverse nature of the uses of the construction.

Technical Reports issued by the Department of Computer Science, Manchester University are available by anonymous ftp from `m1.cs.man.ac.uk`(130.88.13.4) in the directorty `/pub/TR`. The files are stored as PostScript, in compressed form, with the report number as filename. Alternatively, reports are available by post from The computer Library, Department of Computer Science, The University, Manchester M13 9PL, England.

1 Introduction

The original Artin glueing construction [2, IV, 9.5] is a method of reconstructing a given topological space S from an open set D using the relative topologies on D and $S-D$ and some information concerning the boundary between the two sets (the "fringe behaviour"). As with many of the apparently point-sensitive constructions of topology this construction is more easily described in the point-free context, and having done this we are lead to many natural extensions of the method. Thus we work with frames. Given a frame H, let $d \in H$ and consider the lower and upper intervals

$$K = [\bot, d] \quad , \quad L = [d, \top]$$

of H generated by d. Both K and L are themselves frames (in fact quotient frames of H). The problem is to reconstruct H from K and L using the "fringe map"

$$\begin{array}{ccc} K & \xrightarrow{f} & L \\ x & \longmapsto & d \vee (d \supset x). \end{array}$$

I will not explain here why this is the appropriate "fringe behaviour", but notice that f is *not* a frame morphism, merely a $\wedge$-semilattice morphism.

This problem can be given a slightly different perspective.

Suppose we are given any two frames K and L together with a connecting map

$$K \xrightarrow{f} L$$

which is a $\wedge$-semilattice morphism. Let M be the set of all pairs (a, b) where

$$a \in K, b \in L \quad \text{and} \quad b \leq f(a).$$

It is a routine matter to check that M is a frame, where the frame operations (but not the implication operation) are defined pointwise. This frame M is called the *glueing of* K *to* L *along* f. We then find that K and L are, respectively, canonically isomorphic to the lower and upper intervals of M generated by $(\top, \bot)$. Furthermore, when K and L are the given intervals of the frame H, the frame M is canonically isomorphic to H.

This construction can be developed in two slightly different ways.

The more exotic development is due to Wraith [11] who lifted the construction to a 2-categorical setting. He replaced the frames by toposes and the connecting maps by left exact functors to show that the 2-category of toposes has all finite lax co-limits and certain finite (non-lax) co-limits. An account of this is given in [4, pp 107 - 112].

The more elementary development is due to Macnab [7] who considered the problem of reconstructing a frame H from a given finite family

$$H \longrightarrow H(i)$$

($1 \leq i \leq n$) of frame quotients which are jointly monic. Here the construction starts from the frames $H(1), \ldots, H(n)$ and certain connecting maps

$$H(i) \xrightarrow{H(j,i)} H(j)$$

(for $1 \leq i, j \leq n$) which interact in an appropriate way. These maps are again merely $\wedge$-semilattice morphisms.

There are also several other algebraic constructions which can be seen as a use of glueing but were not originally described as such.

The direct topos theoretic analogue of the original Artin construction has been used by Freyd to prove various results in pure type theory. Amongst these are:

- the disjunction property;
- the existential witness property;
- certain conservation results.

An account of these and related results is given in [6, section 20 & 21], and a description of the Freyd version of the glueing construction is given in section 22 of that book. This use of glueing is more intrigueing. It seems to be suggesting that there are fundamental categorical reasons supporting the proof theoretic results, but the precise connection is far from clear.

The nature of the glueing construction is rather curious. First of all the starting diagram seems, to say the least, unpromising and yet it produces the goods. Secondly, the constructed object seems to have some kind of universality which is hard to pin down. It is not at all clear what the construction is attempting to do nor why the result of it should be interesting, but, as we have seen, it does have its uses in some unexpected places.

In this paper I will set down what is, I hope, the beginnings of a coherent account of at least the algebraic version of the construction. I will not attempt to analyse the 2-categorical version for I prefer to get the simpler case sorted out first. The appropriate vehicle for this is poset enriched categories (which are 2-categories without the tedious coherence problems). It is necessary to look at certain laxness properties in these categories.

One benefit of this work is the realization that there are two dual version of the glueing construction, one for each way that strict notions may be relaxed in an enriched category. Only one of these versions seems to have been recognized so far as glueing. However, we now see that, for instance, the analysis given in [9] of filtered co-limits in O-categories involves a glueing construction (in its dual form).

The contents of the remaining sections is as follows.

In section 2 I set down the required background concerning poset enriched categories and laxness in these categories.

The heart of the paper is sections 3,4, and 5. In section 3 I set up the problem which is solved by glueing. This has a local version (the existence of a lax limit for a certain kind of diagram) and a global version (the existence of a right adjoint for a certain "free" functor). In section 4 I look at the details of these problems, in particular I show that the required limit has much more associated structure than might be expected. Finally, in section 5 I show how in certain concrete cases the glueing construction produces the required limit.

Sections 6 and 7 contain a selection of examples of glueing in action. Section 6 gives the more routine applications (some of which have not previously been recognized as glueing constructions). In section 7 I show how the Smyth-Plotkin construction of certain bilimits can be obtained by glueing.

The paper ends with section 8 which contains a selection of problems and questions which may help guide further research.

As is my usual practice, all the diagrams in this paper have been produced using Paul Taylor's customized package.

2 Background requirements

The right kind of ambient categories for our analysis are those which are enriched over posets having appropriate completeness properties.

A poset is, of course, a set Λ together with a carried partial ordering $\leq$ of Λ. The appropriate comparison morphisms

$$\Lambda \xrightarrow{F} \Sigma$$

between two such posets Λ and Σ are the monotone maps, i.e. those functions

$$F : \Lambda \longrightarrow \Sigma$$

for which

$$\psi \leq \phi \;\Rightarrow\; F(\psi) \leq F(\phi)$$

for all $\psi, \phi \in \Lambda$. Given such a poset Λ and a subset $\Phi \subseteq \Lambda$ we write

$$\bigvee \Phi \quad | \quad \bigwedge \Phi \tag{1}$$

for, respectively, the

supremum | infimum

of Φ in Λ, provided such an element exists. We will deal with posets for which certain suprema or infima are guaranteed to exist. There are two, dual, versions of our analysis; one concerning the use of suprema and the other concerning the use of infima. We will refer to these two variants as the

up | down

orientations, respectively. The differences between these versions will become clear when we look at laxness. It is convenient to describe the two variants in parallel.

A minimal kind of completeness for a poset Λ is the existence of

$$\bigvee\emptyset \quad | \quad \bigwedge\emptyset$$

i.e. the existence of a

$$\text{bottom } \perp \quad | \quad \text{top } \top$$

of Λ. More generally we may require the existence of (1) for all finite Φ. In this case Λ becomes, respectively, a

$$\vee-\text{semilattice} \quad | \quad \wedge-\text{semilattice}.$$

Notice that these two completeness properties are independent; we can have one without the other. Note also that we do not expect a comparison morphism between posets to respect any completeness properties of the source and target. The most severe completeness property of a poset is the existence of the sup/inf of (1) for all subsets Φ. In this case each of these completeness properties implies the other, however, as will become apparent later, we do want to treat the two cases separately. In between these two extremes of completeness there is a whole spectrum of completeness properties where in each case we require the existence of (1) for all Φ of size no bigger than a given cardinality. (I have explicitly described here the 1, $\aleph_0$, and the ∞ cases.) Recall that a subset Φ of a poset Λ is

$$\text{up} \quad | \quad \text{down}$$

directed if it is non-empty and for each $\theta, \psi \in \Phi$ there is some $\phi \in \Phi$ with

$$\theta, \psi \leq \phi \quad | \quad \phi \leq \psi, \theta.$$

A different kind of completeness requires the existence of (1) for all appropriately directed subsets Φ. This notion may be specialized further by restricting the directed sets considered to (for example)

$$\text{ascending} \quad | \quad \text{descending}$$

ω-chains. (It is a curious fact that although both versions of the cardinality completeness properties occur naturally, only the up version of directed completeness seems to occur in practice.)

At various times we will have in mind a notion of completeness such as the several examples given above. It is not necessary to delimit here the precise nature of such notions, (although this could be done, perhaps, using the subset systems of [12]). However, it is useful to introduce some terminology to aid the exposition. We use that informal notion of a *nice* subset of a poset. The meaning of 'nice' depends on and determines the current notion of completeness. We will not make this notion more precise, but it could mean for instance, having restricted cardinality, or being appropriately directed. Given a current meaning of 'nice' we say a poset is *nicely complete* if for each if its nice subsets the

$$\text{supremum} \quad | \quad \text{infimum}$$

exists, as appropriate.

A category Cat is enriched over posets if for each pair A, B of Cat-objects, the arrow set

$$\mathsf{Cat}[A, B]$$

is partially ordered and composition is monotone in the sense that

$$\left.\begin{array}{l} g \leq f \\ \\ k \leq h \end{array}\right\} \Rightarrow \ kg \leq hf$$

holds for all couples of parallel pairs

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] & C \end{tikzcd}$$

of arrows. These are the only kind of enriched categories we need, so we simply refer to them as 'enriched categories'.

As above let 'nice' refer to a kind of subset of posets which determines a completeness property in one of the two orientations. Let Cat be an enriched category. We say that Cat is *nicely enriched* if each arrow set is nicely complete, and composition of arrows respects this completeness in the sense that for each pair of arrows

$$A \xrightarrow{f} B \qquad , \qquad C \xrightarrow{h} D$$

and nice subset G of $\mathsf{Cat}[B, C]$ the subset

$$K \ = \ \{hgf \mid g \in G\}$$

of $\mathsf{Cat}[A, D]$ is also nice and the appropriate identity

$$h(\bigvee G)f \ = \ \bigvee K \quad \mid \quad h(\bigwedge G)f \ = \ \bigwedge K$$

holds (depending on the current orientation). Our analysis will take place in such a nicely enriched category with a determined orientation. It is important that you know which of these two orientations is operative at any one time; some of the details of the laxness calculations are dependent on this.

Note that any enriched category Cat can be re-enriched (to give a different enriched category Cat^{co}) by reversing the comparison convention on arrows. This will change any niceness properties that Cat has, in particular it will change an up orientation to a down orientation, and conversely. This indicates that, in theory, only one of the orientations is necessary; however, there are naturally occurring examples in both orientations and it is as well to have these alternatives set down if only for benchmark purposes.

The simplest example of an enriched category is the category Pos of posets and monotone maps with the enrichment given by the pointwise comparison on arrows. This category has no niceness whatsoever. It may appear that for this example we need not distinguish between the up and the down orientations. However, even here laxness considerations will force us to recognize these two distinct cases. Let

$$\mathsf{Slt}^{\vee} \quad | \quad \mathsf{Slt}^{\wedge}$$

be, respectively, the categories of

$$\vee-\text{semilattices} \quad | \quad \wedge-\text{semilattices}$$

and attribute preserving morphisms. Both of these are enriched using the pointwise comparison. Furthermore, both are nicely enriched with nice meaning finite in the

$$\text{up} \quad | \quad \text{down}$$

orientation, respectively. (If you have never done this before, you should check these assertions and note why each step in the calculation holds.) The categories

$$\mathsf{Sup} \quad | \quad \mathsf{Inf}$$

of

$$\bigvee-\text{semilattices} \quad | \quad \bigwedge-\text{semilattices}$$

and associated morphisms are, using the pointwise comparisons, arbitrarily enriched in the

$$\text{up} \quad | \quad \text{down}$$

orientation. (Again you should check these assertions.) More generally, the arbitrarily enriched categories in the up orientation are precisely the quantaloids of [1] Various categories of domains are directedly (or perhaps just ω-chain) enriched in the up orientation. The general properties of these categories are precisely captured by the O-categories of Smyth and Plotkin [9] and Wand [10]. I will have more to say about these in section 7.

In any enriched category Cat an *adjoint pair* is a pair of Cat-arrows

$$A \underset{f_*}{\overset{f^*}{\rightleftarrows}} B$$

for which

$$1_A \le f_* f^* \quad , \quad f^* f_* \le 1_B .$$

By convention we call f^* the left component of the pair and f_* the right component of the pair. We sometimes write

$$A \xrightarrow{f^* \dashv f_*} B$$

to indicate such an adjoint pair and this points the pair in the direction of the left component. Note that these adjoint pairs need not be adjunctions in the conventional sense (since the source and target need not be categories). However, these pairs do have some properties in common with adjunctions. For instance, both the identities

$$f^* f_* f^* \;=\; f^* \quad , \quad f_* f^* f_* \;=\; f_*$$

hold. Also for each couple

$$f^* \dashv f_* \quad , \quad g^* \dashv g_*$$

of adjoint pairs

$$A \longrightarrow B$$

the equivalence

$$f^* \leq g^* \;\Leftrightarrow\; g_* \leq f_*$$

holds. In particular, each component of an adjoint pair uniquely determines the other. These adjoint pairs compose in the obvious way to give us the category $\mathsf{Cat}^{\dashv}$ of Cat-objects and adjoints pairs. Note that there are two forgetful functors

$$\mathsf{Cat}^{\dashv} \longrightarrow \mathsf{Cat}$$

formed by taking the left and the right component of an adjoint pair, respectively. The second of these is contravariant.

In a strict setting the algebra of arrow composition in a category Cat is straight forward. Given a triangle

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & C \\ & B \arrow[ur, "g"'] & \end{tikzcd}$$

of Cat-arrows all that we are interested in is whether or not

$$h \;=\; gf.$$

When this holds we say the triangle commutes. In the enriched case we may ask for a weaker relationship to hold between h and gf, namely that they are comparable. This gives us two relaxed version of commutation depending on the order of comparison of the two arrows.

In an enriched category Cat we say the above triangle

up commutes $\quad|\quad$ down commutes

if

$$h \geq gf \quad | \quad h \leq gf$$

holds, respectively. As the theory develops we will see other lax notions which come in dual pairs, an up version and a down version. To help you distinguish between these I suggest you think of the passage from gf to h as a kind of normalization. Thus, in the up version this passage proceeds in an upwards direction, and in the down version this passage proceeds in a downwards direction.

3 The categorical problem

Let Cat be a fixed poset enriched category with an assumed orientation (up or down). We do not assume that Cat has any particular niceness properties. We wish to compute the limit of a certain lax diagram in Cat based on a complete graph. Thus let I be a fixed set (with elements $i, j, k, \ldots, r, s, t, \ldots$). We convert this into the *indiscrete category* on I by introducing precisely one arrow

$$i \xrightarrow{(j,i)} j$$

for each $i, j \in I$. In particular (i,i) is the identity arrow on the object i, and for all $i, j, k \in I$ we have

$$(k,j)(j,i) = (k,i)$$

that is, the triangle

$$\begin{tikzcd} i \arrow[rr, "{(k,i)}"] \arrow[dr, "{(j,i)}"'] & & k \\ & j \arrow[ur, "{(k,j)}"'] & \end{tikzcd}$$

commutes. We will use this indiscrete category to index the diagram in Cat whose limit we require

3.1 DEFINITION. An I-indexed *web* in Cat is a lax functor

$$I \xrightarrow{\mathfrak{A}} \mathsf{Cat}$$

from the indiscrete category I to Cat ■

Later I will give an explicit description of the various components of a web. In particular I will indicate where the laxness is relevant.

Two webs $\mathfrak{B}$ and $\mathfrak{A}$ (of the same kind) are compared using a lax transformation

$$\mathfrak{B} \longrightarrow \mathfrak{A}.$$

It is routine to check that these transformations are closed under composition, and hence we obtain a category. Two transformations

$$\mathfrak{B} \rightrightarrows \mathfrak{A}$$

can be compared using the pointwise comparison, and this provides a poset enrichment of the category. Again the details of these notions will be set out later.

3.2 DEFINITION. Let CatI be the enriched category of I-indexed webs in Cat and lax transformations (relative to the assumed orientation). ■

Each object A (of Cat) gives us a particularly simple web, namely, the functor which assigns A to each index $i \in I$ and the identity arrow 1_A to each arrow (j, i) of I. This is actually a strict functor

$$I \xrightarrow{\mathcal{F}(A)} \mathsf{Cat}$$

so that $\mathcal{F}(A)$ is a web no matter which orientation of Cat is current. We call $\mathcal{F}(A)$ the *constant web* with value A. It is easy to check that each arrow

$$B \longrightarrow A$$

(of Cat) induces a transformation

$$\mathcal{F}(B) \longrightarrow \mathcal{F}(A)$$

and hence we have a (strict) functor

$$\mathsf{Cat} \xrightarrow{\mathcal{F}} \mathsf{Catl}.$$

This functor is monotone in the sense that for all objects A, B the assignment

$$\mathsf{Cat}[B, A] \longrightarrow \mathsf{Catl}[\mathcal{F}(B), \mathcal{F}(A)]$$

is a poset morphism.

We are interested in answering certain questions concerning the functor $\mathcal{F}$.

(1) When does $\mathcal{F}$ have a right adjoint?

Of course, because of the enrichment and laxness of the situation, it is not at absolutely clear what we mean by a 'right adjoint'. Nevertheless, we will see that, provided the enrichment of Cat is sufficiently nice, the glueing construction $\mathcal{G}$ is a right adjoint to $\mathcal{F}$.

(2) For a given web $\mathfrak{A}$, when does $\mathfrak{A}$ have a limit?

Again it is not absolutely clear what a 'limit' should be. However, we should view (2) as a local version of (1) (in the sense that $\mathcal{F}$ has a right adjoint precisely when each web $\mathfrak{A}$ has a limit). We will find that the glueing construction applied to $\mathfrak{A}$ gives the required limit.

To answer these questions we need to investigate how an object A (of Cat) and a web $\mathfrak{A}$ can interact. Notice that the object A gives us a (strict) contravariant functor

$$\mathsf{Cat} \longrightarrow \mathsf{Pos}$$

namely, the hom-functor

$$\mathsf{Cat}[-, A].$$

Similarly, the web $\mathfrak{A}$ gives us a parallel functor, namely, the restricted hom-functor

$$\mathsf{Catl}[\mathcal{F}(-), \mathfrak{A}].$$

We can ask for a comparison between these functors.

(3) Given a web $\mathfrak{A}$, when is there an object A such that the above two functors are naturally isomorphic?

Such an object A is the limit of $\mathfrak{A}$ and, as you might expect, will be given by the glueing construction.

For the two hom-functors to be isomorphic we at least need a (strict) natural transformation

$$\mathsf{Cat}[-, A] \longrightarrow \mathsf{Catl}[\mathcal{F}(-), \mathfrak{A}]$$

(which is then required to have appropriate extra properties). In the usual way we see that such a transformation is completely determined by the image of $\mathbf{1}_A$ under the assignment

$$\mathsf{Cat}[A, A] \longrightarrow \mathsf{Catl}[\mathcal{F}(A), \mathfrak{A}]$$

thus we need to study the lax transformations

$$\mathcal{F}(A) \longrightarrow \mathfrak{A}.$$

These are nothing more than the cones

$$A \longrightarrow \mathfrak{A}$$

of $\mathfrak{A}$, and they are the gadgets that we take as the starting point of our analysis.

4 Some details

We have already fixed both the enriched category Cat with an assumed orientation, and the indexing set (category) I. We now fixed the web $\mathfrak{A}$ and look for a limit of $\mathfrak{A}$ (in the sense of the previous section). To do that we need to spell out the details of the components of $\mathfrak{A}$ and introduce some notation for these components.

With all the definitions and results in this paper there are two versions, an up version and a down version. These differ only in a few crucial places to allow a parallel development of the two versions as in the next definition.

4.1 DEFINITION. A *web* $\mathfrak{A}$ in the

up | down

orientation is

- an object $A(i)$, for each $i \in I$,
- an arrow $A(i) \xrightarrow{A(j,i)} A(j)$, for each pair $i, j \in I$

where the comparisons

$$\begin{array}{r|l} A(i,i) \geq \mathbf{1}_{A(i)} & A(i,i) \leq \mathbf{1}_{A(i)} \\ A(k,i) \geq A(k,j)A(j,i) & A(k,i) \leq A(k,j)A(j,i) \end{array}$$

respectively, hold for all $i, j, k \in I$. The objects $A(i)$ are called the *nodes* of $\mathfrak{A}$, and the arrows $A(j,i)$ are called the *edges* of $\mathfrak{A}$ ■

This definition contains alternatives, indicated by

$$\cdots | \cdots$$

at two points. When reading the definition you should make a consistent choice of these alternatives. For example, to extract the definition of an *up web* you make the left hand choice at the two positions where there are alternatives. The notion of a *down web* follows by making the right hand choice in the two positions. The notions obtained by matching the left hand choice in one position with the right hand choice in the other position are not of interest in this paper.

In a similar fashion we can detail the components and properties of a lax transformation

4.2 DEFINITION. Let $\mathfrak{B}$ and $\mathfrak{A}$ be a pair of webs of the same kind. A *lax transformation*

$$\mathfrak{B} \xrightarrow{\phi} \mathfrak{A}$$

is a family of arrows

$$B(i) \xrightarrow{\phi(i)} A(i)$$

$(i \in I)$ such that for each $i, j \in I$ the square

$$\begin{tikzcd} B(i) \arrow[r, "\phi(i)"] \arrow[d, "B(j,i)"'] & A(i) \arrow[d, "A(j,i)"] \\ B(j) \arrow[r, "\phi(j)"'] & A(j) \end{tikzcd}$$

lax commutes, i.e. satisfies

$$\phi(j)B(j,i) \ \geq \ A(j,i)\phi(i) \quad \Big| \quad \phi(j)B(j,i) \ \leq \ A(j,i)\phi(i)$$

for the

up | down

orientation, respectively. ■

We are concerned with finding a limit of the given web $\mathfrak{A}$. This will consist of an object together with a wedge of arrows emanating from the object and going to the nodes of $\mathfrak{A}$. These arrows must concur with the edges of $\mathfrak{A}$ (in some appropriate sense), and enjoy a certain universality amongst all such concurring wedges. If, for the time being, we ignore this universality then we obtain the notion of a cone for the web. We may then dualize this to obtain the notion of a co-cone for the web. Both of these are of interest to us.

Thus, a cone or co-cone for $\mathfrak{A}$ is an object A together with a family of arrows attaching A to the nodes of $\mathfrak{A}$. For a cone these arrows have A as a common source, and for a co-cone they have A as a common target. Two such arrows can be compared by composing one of them with the appropriate edge of $\mathfrak{A}$. This gives us a triangle which is required to lax commute. We find that for the cones and co-cones we are concerned with, these attaching arrows form the components of one side of a family of adjunctions between A and the nodes of $\mathfrak{A}$. I will use a notation to indicate this.

The following definition again offers several alternatives such as up/down, cone/co-cone, left/right etc. These choices must be fitted together in a consistent way to produce just four notions.

4.3 DEFINITION. Let $\mathfrak{A}$ be, respectively, a web $\mathfrak{A}$ in the

up	down
up	down.

orientation. A

co-cone	cone
cone	co-cone

for $\mathfrak{A}$ is an object A together with a family of arrows

$$\begin{array}{c|c} A \xleftarrow{\;f^i\;} A(i) & A \xrightarrow{\;f^i\;} A(i) \\ \hline A \xrightarrow[\;f_i\;]{} A(i) & A \xleftarrow[\;f_i\;]{} A(i) \end{array}$$

(for $i \in I$) such that

$$\begin{array}{c|c} f^i \geq f^j A(j,i) & f^j \leq A(j,i) f^i \\ \hline f_j \geq A(j,i) f_i & f_i \leq f_j A(j,i) \end{array}$$

holds for all $i, j \in I$. ■

You should note the position of the labels attached to the structuring arrows. These have been chosen with some care. At times it will be convenient to use an abbreviated notation. Thus we write

$A \xleftarrow{f^\bullet} \mathfrak{A}$	$A \xrightarrow{f^\bullet} \mathfrak{A}$
$A \xrightarrow[f_\bullet]{} \mathfrak{A}$	$A \xleftarrow[f_\bullet]{} \mathfrak{A}$

to indicate that $\mathfrak{A}$ is

an up web	a down web

and A is the vertex of a

co-cone	cone
cone	co-cone

for $\mathfrak{A}$ with the given structuring arrows.

Each web has embedded in it several cones and co-cones.

4.4 LEMMA. *Let $\mathfrak{A}$ be a web and let r be a fixed index. Then the family of arrows*

$$A(r) \xrightarrow{A(i,r)} A(i)$$

(for $i \in I$) forms a cone with vertex $A(r)$ and the family of arrows

$$A(r) \xleftarrow{A(r,i)} A(i)$$

forms a co-cone with vertex $A(r)$.

Proof. For instance, in the up case we have

$$A(j,r) \geq A(j,i)A(i,r)$$

(for $i,j \in I$) which shows that the arrows $A(\bullet,r)$ form a cone. ■

You should check that in both orientations a cone

$$A \longrightarrow \mathfrak{A}$$

is precisely the same as a lax transformation

$$\mathcal{F}(A) \longrightarrow \mathfrak{A}.$$

However, co-cones do not have a corresponding description.

It is easy to check that such a cone

$$A \longrightarrow \mathfrak{A}$$

of the web $\mathfrak{A}$ induces natural transformation

$$\mathsf{Cat}[-,A] \longrightarrow \mathsf{Catl}[\mathcal{F}(-),\mathfrak{A}]$$

whose action is given by composition. We say the cone is *limiting* if this transformation is an isomorphism (i.e. a poset isomorphism). This notion is the composite of two distinct properties, namely that for each object B the induced poset morphism

$$\mathsf{Cat}[B, A] \longrightarrow \mathsf{Catl}[\mathcal{F}(B), \mathfrak{A}]$$

is *surjective* and an *embedding*. These two properties can be applied independently to cones. It is also useful to have analogous properties for co-cones. The details of these are as follows.

4.5 DEFINITION. Let $\mathfrak{A}$ be, respectively, a web in the

$$\text{up} \quad | \quad \text{down}$$

orientation, and let

$$\begin{array}{c|c} A \xleftarrow{\ f^\bullet\ } \mathfrak{A} & A \xrightarrow{\ f^\bullet\ } \mathfrak{A} \\ \hline A \xrightarrow[\ f_\bullet\ ]{} \mathfrak{A} & A \xleftarrow[\ f_\bullet\ ]{} \mathfrak{A} \end{array}$$

be, respectively, a

$$\begin{array}{c|c} \text{co-cone} & \text{cone} \\ \hline \text{cone} & \text{co-cone} \end{array}$$

for $\mathfrak{A}$. We refer to this as 'the gadget'.

(a) The gadget is *generating* is for each other gadget

$$\begin{array}{c|c} B \xleftarrow{\ \phi^\bullet\ } \mathfrak{A} & B \xrightarrow{\ \phi^\bullet\ } \mathfrak{A} \\ \hline B \xrightarrow[\ \phi_\bullet\ ]{} \mathfrak{A} & B \xleftarrow[\ \phi_\bullet\ ]{} \mathfrak{A} \end{array}$$

of the same kind, there is at least one factoring of this gadget through f, i.e. at least one arrow

$$\begin{array}{c|c} B \xleftarrow{\ \phi\ } A & B \xrightarrow{\ \phi\ } A \\ \hline B \xrightarrow[\ \phi\ ]{} A & B \xleftarrow[\ \phi\ ]{} A \end{array}$$

with

$$\begin{array}{c|c} \phi^i = \phi f^i & \phi^i = f^i \phi \\ \hline \phi_i = f_i \phi & \phi_i = \phi f_i \end{array}$$

for all $i \in I$.

(b) The gadget is *separating* if for each pair of parallel arrows

$$\begin{array}{c|c} B \underset{\psi}{\overset{\phi}{\leftleftarrows}} A & B \underset{\psi}{\overset{\phi}{\rightrightarrows}} A \\ \hline B \underset{\psi}{\overset{\phi}{\rightrightarrows}} A & B \underset{\psi}{\overset{\phi}{\leftleftarrows}} A \end{array}$$

if, respectively,

$$\begin{array}{c|c} (\forall i \in I)[\phi f^i \ \geq \ \psi f^i] & (\forall i \in I)[f^i \phi \ \geq \ f^i \psi] \\ \hline (\forall i \in I)[f_i \phi \ \geq \ f_i \psi] & (\forall i \in I)[\phi f_i \ \geq \ \psi f_i] \end{array}$$

then $\phi \geq \psi$. ■

By definition, a limit of the web $\mathfrak{A}$ is a cone

$$A \longrightarrow \mathfrak{A}$$

which is both generating and separating. It routine to show that any two limiting cones are canonically isomorphic. What is not so obvious is that the limiting properties ensure that there is also a co-cone

$$A \longleftarrow \mathfrak{A}$$

where the two families of arrows combine in a nice way to produce a family of adjunctions. By extracting the pertinent properties of this pair of families we obtain the following central notion.

4.6 DEFINITION. Let $\mathfrak{A}$ be, respectively, a web in the up/down orientation. A *spider* for $\mathfrak{A}$ is a family

$$\begin{array}{c|c} A \underset{f_i}{\overset{f^i}{\leftrightarrows}} A(i) & A \underset{f_i}{\overset{f^i}{\rightleftarrows}} A(i) \end{array}$$

of adjoint pairs $f^i \dashv f_i$ of arrows (for $i \in I$) satisfying

$$\begin{array}{c|c} f_t f^s = A(t,s) & f^t f_s = A(t,s) \\ \bigvee\{f^i f_i \mid i \in I\} = 1_A & \bigwedge\{f_i f^i \mid i \in I\} = 1_A \end{array}$$

for all $s, t \in I$. ■

You should note that part of the property of being a spider is that the possibly infinite supremum/infimum in the definition must exist (and be 1_A). It is not guaranteed to exist by any assumed niceness properties of Cat.

It is routine to check that each spider of a web provides both a cone and a co-cone. For instance, in the down case

$$A \xrightarrow{f^\bullet \dashv f_\bullet} \mathfrak{A}$$

we have, for each $i, j \in I$,

$$1_A \leq f_i f^i \quad , \quad f^j f_i = A(j, i)$$

so that

$$A(j, i) f^i = f^j f_i f^i \leq fju$$

as required to show that $f^\bullet$ is a cone.

The next result explains why we are interested in spiders.

4.7 THEOREM. *Let $\mathfrak{A}$ be a web and let*

$$A \longrightarrow \mathfrak{A}$$

be a limiting cone for $\mathfrak{A}$. Then these arrows are the components of one side of a family of adjunctions which structure A as a spider for $\mathfrak{A}$. In the up/down case they are the right/left components, respectively.

Proof. Let us look at the up case. (The down case follows by a dual argument.) Thus we are given a generating and separating cone

$$A \xrightarrow[f_\bullet]{} \mathfrak{A}$$

and we must produce a certain co-cone

$$A \xleftarrow{f^\bullet} \mathfrak{A}.$$

By Lemma 4.4, for each $s \in I$ we have a cone

$$A(s) \xrightarrow{A(\bullet, s)} \mathfrak{A}$$

for $\mathfrak{A}$, and hence the generating property gives an arrow

$$A \xleftarrow{f^s} A(s)$$

such that

$$f_i f^s \;=\; A(i, s)$$

for all $i \in I$. These are the co-cone structuring arrows.

Next, for all $i, j \in I$, we have

$$f_j f^i f_i \ = \ A(j,i) f_i \ \leq \ f_j \ = \ f_j \mathbf{1}_A$$

so that, by the separation property, we have $f^i f_i \ \leq \ \mathbf{1}_A$.

Finally, consider any other endo-arrow ϕ of A with $f^i f_i \ \leq \ \phi$ for each $i \in I$. Then

$$f_i \phi \ \geq \ f_i f^i f_i \ = \ A(i,i) f_i \ \geq \ \mathbf{1} A(i) f_i \ = \ f_i \ = \ f_i \mathbf{1}_A$$

so that, again by the separation property, $\phi \ \geq \ \mathbf{1}_A$. This shows that

$$\bigvee \{ f^i f_i \mid i \in I \}$$

exists and is $\mathbf{1}_A$, as required. ■

This result shows that, if it exists, the limit of a web can be found amongst the associated spiders. In general a web may have many such spiders, but in one important case there is a uniques one. We say that Cat is I-nice if within arrows sets all I-indexed suprema/infima exist (depending on the current orientation).

4.8 THEOREM. *Suppose* Cat *is I-nice and let $\mathfrak{A}$ be a web. Then each spider for $\mathfrak{A}$ provides a limiting cone for $\mathfrak{A}$. Furthermore, the associated co-cone of the spider is both generating and separating.*

Proof. We give the proof for the up case. Thus let

$$A \underset{f_\bullet}{\overset{f^\bullet}{\leftrightarrows}} \mathfrak{A}$$

be the given spider; we must show that the cone $f_\bullet$ and the co-cone $f^\bullet$ are both generating and separating. Again the proofs of these are similar.

For instance let

$$B \xleftarrow{\ \phi^\bullet\ } \mathfrak{A}$$

be any co-cone for $\mathfrak{A}$ and consider

$$\phi \ = \ \bigvee \{ \phi^i f_i \mid i \in I \}.$$

(This exists because Cat is I-nice.) Then, for each $s \in I$, we have

$$\phi f^s \ = \ \bigvee \{ \phi^i f_i f^s \mid i \in I \} \ = \ \bigvee \{ \phi^i A(i,s) \mid i \in I \} \ \leq \ \phi^s$$

and

$$\phi f^s \ \geq \ \phi^s A(s,s) \ \geq \ \phi^s$$

so that

$$\phi f^s \;=\; \phi^s.$$

This shows that $f^\bullet$ is a generating family.

Next let

$$\begin{tikzcd} B & A \arrow[l, shift right, "\phi"'] \arrow[l, shift left, "\psi"] \end{tikzcd}$$

be a pair of arrows for which

$$(\forall i \in I)[\phi f^i \;\geq\; \psi f^i].$$

Then

$$\psi \;=\; \psi 1_A \;=\; \bigvee\{\psi f^i f_s \mid i \in I\} \;\leq\; \bigvee\{\phi f^i f_s \mid i \in I\} \;=\; \phi 1_A \;=\; \phi$$

which shows that the family $f^\bullet$ is separating. ■

5 The glueing construction

By definition, the limit A of a web $\mathfrak{A}$ is a spider for which the induced natural transformation

$$\mathsf{Cat}[-, A] \longrightarrow \mathsf{Catl}[\mathcal{F}(-), \mathfrak{A}]$$

is an isomorphism. As yet I have said nothing about the existence of such limits, and it is now time to consider this aspect. This, of course, is the job of the glueing construction.

Suppose the objects of Cat are structured posets and that the enrichment of Cat is given by the pointwise comparison of arrows. Suppose also that Cat has a separator (sometimes called a generator), that is, an object S such that for each object A, the elements of A are in bijective correspondence with the Cat-arrows

$$\mathsf{S} \longrightarrow A.$$

In this case the required limit A satisfies

$$A \;\cong\; \mathsf{Cat}[\mathsf{S}, A] \;\cong\; \mathsf{Catl}[\mathcal{F}(\mathsf{S}), \mathfrak{A}].$$

This means that each element of A is a family $(a(i) \mid i \in I)$ of elements of the nodes of $\mathfrak{A}$ which fit together to form a lax transformation

$$\mathcal{F}(\mathsf{S}) \longrightarrow \mathfrak{A}.$$

Unravelling this we arrive at the notion of a *thread*.

Given a web $\mathfrak{A}$ let

$$\mathcal{A} \;=\; (A(i) \mid i \in I)$$

be the indexed family of its nodes. Recall that a choice function for this family is a function

$$a : I \longrightarrow \bigcup \mathcal{A}$$

such that $a(i) \in A(i)$ for all $i \in I$. We look at those choice functions which are compatible with the web arrows.

5.1 DEFINITION. For each web $\mathfrak{A}$ in the up/down orientation a *thread* of $\mathfrak{A}$ is a choice function $a(\cdot)$ of $\mathcal{A}$ such that

$$a(j) \;\geq\; A(j,i)(a(i)) \quad \mid \quad a(j) \;\leq\; A(j,i)(a(i))$$

holds, respectively, for all $i, j \in I$. ■

Let A be the set of all threads of $\mathfrak{A}$. Clearly A is a poset under the pointwise comparison. In all the concrete cases I have looked at it is easy to see that, in fact, A is a Cat-object. Similarly, for each $i \in I$, the evaluation map

$$\begin{array}{ccc} A & \longrightarrow & A(i) \\ a & \longmapsto & a(i) \end{array}$$

is a Cat-arrow. Now comes the crucial observation.

5.2 THEOREM. *Let A be the object of threads of the web $\mathfrak{A}$. Then the evaluation arrows structure A as the limit of $\mathfrak{A}$.*

Proof. (This proof contains some hidden assumptions concerning the nature of the category Cat.)

Let us look at the up case (and leave the down case as an exercise). Thus, using the standard notation, we have a family

$$A \xrightarrow[f_\bullet]{} \mathfrak{A}$$

of arrows where

$$f_i(a) \;=\; a(i)$$

for each $a \in A$ and $i \in I$. Using the thread property, for each $i, j \in I$ and $a \in I$ we have

$$A(j,i)f_i(a) \;=\; A(j,i)(a(i)) \;\leq\; a(j) \;=\; f_j(a)$$

so that $f_\bullet$ is a lax cone for $\mathfrak{A}$.

To verify the generating property consider any other lax cone

$$B \xrightarrow[\phi_\bullet]{} \mathfrak{A}$$

of $\mathfrak{A}$. For each $b \in B$ let $\phi(b)$ be the choice function of $\mathcal{A}$ given by

$$\phi(b)(i) \;=\; \phi_i(b).$$

Then, for each $i, j \in I$ we have

$$A(j,i)(\phi(b)(i)) \;=\; A(j,i)\phi_i(b) \;\leq\; \phi_j(b) \;=\; \phi(b)(j)$$

which shows that $\phi(b)$ is a thread of $\mathfrak{A}$.

Thus we have a map

$$B \xrightarrow[\phi]{} A$$

which for all considered concrete cases is a Cat-arrow. By construction this arrow satisfies $f_\bullet \phi \;=\; \phi_\bullet$ which gives the required genericity.

Finally, to verify the separation property, consider any parallel pair

$$B \underset{\psi}{\overset{\phi}{\rightrightarrows}} A$$

of arrows such that $f_i\psi \;\leq\; f_i\phi$ for all $i \in I$. Then, for each $b \in B$ and $i \in I$ we have

$$\psi(b) \;=\; f_i\psi(b) \;\leq\; f_i\phi(b) \;=\; \phi(b)$$

so that $\psi(b) \leq \phi(b)$ and hence $\psi \leq \phi$, as required. ■

You may like to check that, in the up case, the co-cone arrows

$$A \xleftarrow{f^\bullet} \mathfrak{A}$$

are given by

$$f^i(x)(j) \;=\; A(j,i)(x)$$

for each $i, j \in I$ and $x \in A(i)$. The down version is similar.

5.3 DEFINITION. For each web $\mathfrak{A}$ the spider of all threads of $\mathfrak{A}$ is called the *glued version* of $\mathfrak{A}$, and is denoted by $\mathcal{G}(\mathfrak{A})$. ■

By now we have enough information to give several different proofs of the following result. I leave it as an exercise for you to find at least one such proof.

5.4 THEOREM. *The construction $\mathcal{G}(\cdot)$ is a monotone functor and the pair*

$$\mathsf{Cat} \underset{\mathcal{G}(\cdot)}{\overset{\mathcal{F}(\cdot)}{\rightleftarrows}} \mathsf{CatI}$$

form an adjunction (with $\mathcal{F} \dashv \mathcal{G}$).

6 Some examples

In this section I give some examples of webs and spiders. All of these have appeared in the literature, but not all of them have been previously recognized as applications of the glueing construction.

6.1 1-webs

The simplest example of a web is the one based on a 1-element index set. Such a web consists of an object Z together with an endo-arrow ζ of Z for which

$$\begin{array}{c|c} 1_Z \leq \zeta & \zeta \leq 1_Z \\ \zeta^2 \leq \zeta & \zeta \leq \zeta^2 \end{array}$$

depending on which of the up/down orientations is current. Since composition is monotone these conditions ensure that ζ is idempotent. A spider for this web is an object A together with a pair of arrows

$$Z \xrightarrow{f} A \xrightarrow{g} Z$$

such that

$$f \dashv g \quad | \quad g \dashv f$$

and, in both orientations,

$$gf = \zeta \qquad fg = 1_A.$$

The spider is the object of fixed points of ζ, and the glueing construction picks out precisely these elements.

6.2 2-webs

The most general case of a 2-element index web consists of a pair of objects K and L together with two arrows

$$\begin{tikzcd} K \arrow[r, shift left, "f"] & L \arrow[l, shift left, "g"] \end{tikzcd}$$

and endo-arrows κ of K and λ of L satisfying

$$\kappa^2 = \kappa \quad , \quad \lambda^2 = \lambda$$

$$f\kappa = f = \lambda f$$

$$\kappa g = g = g\lambda$$

and

$$\begin{array}{c|c} 1_K \leq \kappa \ , \ 1_L \leq \lambda & \kappa \leq 1_K \ , \ \lambda \leq 1_L \\ gf \leq \kappa \ , \ fg \leq \lambda & \kappa \leq gf \ , \ \lambda \leq fg \end{array}$$

depending on which of the up/down orientation is current. The equalities here are consequences of the required comparisons combined with the monotonicity of composition. When K and L are objects of a category of structured posets, the glued version of this web is based on the set of all those pairs $(a,b) \in K \times L$ where

$$\kappa(a) = b \quad , \quad \lambda(b) = b$$

and certain other comparisons hold.

6.3 Artin webs

In many examples of 2-webs not all of the arrows f, g, κ, λ are visible. In particular we often have $\kappa = 1_K$ and $\lambda = 1_L$, and then only the comparisons

$$gf \leq 1_K \ , \ fg \leq 1_L \quad | \quad 1_K \leq gf \ , \ 1_L \leq fg$$

are necessary. To satisfy these suppose that the arrow set $\mathsf{Cat}[L, K]$ has a

$$\text{bottom } \perp \quad | \quad \text{top } \top$$

and let g be this extreme arrow. Then f can be any arrow $K \longrightarrow L$ whatsoever. The down version of this is the situation for the original Artin glueing construction. The arrow f is then the fringe map.

6.4 Capping a frame

For a filter ∇ of a frame H let

$$H \xrightarrow{\ f\ } 2$$

be the arrow given by

$$f(x) = \begin{cases} 1 & \text{if } x \in \nabla \\ \\ 0 & \text{if } x \notin \nabla. \end{cases}$$

Using this as a fringe map we find that the glued version is the set of all pairs

$$\begin{array}{ll} (a, 1) & \text{for } a \in \nabla \\ (a, 0) & \text{for } a \in H \end{array}$$

with the obvious structure. Thus we obtain a copy of H with the filter ∇ covered by a cap. In the particular case where ∇ is the filter of co-compact elements of H, this is the 1-point compactification of H. An early use of this construction can be found in [8].

6.5 Certain pullbacks

Let Cat be at least finitely enriched in the down orientation and consider a co-wedge

$$\begin{tikzcd} & L \arrow[d, "l^* \dashv l_*"] \\ K \arrow[r, "k^* \dashv k_*"'] & M \end{tikzcd}$$

of adjoint pairs of Cat-arrows (i.e. a co-wedge in $\mathsf{Cat}^{\dashv}$). For reasons I don't understand we need to assume that

$$k^*k_* \;=\; 1_M \;=\; l^*l_*. \tag{2}$$

Set

$$f = l_*k^* \quad , \quad g = k_*l^*$$

to produce a pair of maps

$$\begin{tikzcd} K \arrow[r, shift left, "f"] & L. \arrow[l, shift left, "g"] \end{tikzcd}$$

Note that

$$gf \;=\; k_*l^*l_*k^* \;=\; k_*k^* \;\geq\; 1_K \quad , \quad fg \;=\; l_*k^*k_*l^* \;=\; l_*l^* \;\geq\; 1_L$$

where the central step uses (2). (This is the only place where (2) is used.) Thus these arrows with the identity arrows give a down web on the nodes K and L.

Now let

$$\begin{tikzcd} K \arrow[r, shift right, "f_*"'] & A \arrow[l, shift right, "f^*"'] \arrow[r, shift left, "g^*"] & L \arrow[l, shift left, "g_*"] \end{tikzcd}$$

be a spider for this web. These arrows satisfy

$$\begin{gathered} f^*f_* = 1_K \quad , \quad g^*g_* = 1_L \\ g^*f_* = f \quad , \quad f^*g_* = g \\ f_*f^* \wedge g_*g^* \;=\; 1_A \end{gathered}$$

and it is an easy exercise to show that they provide:

(a) the pullback in Cat of the arrows k^*, l^*;

(b) the pushout in Cat of the arrows k_*, l_*;

(a) the pullback in $\mathsf{Cat}^{\dashv}$ of the arrows $k^* \dashv k_*$ and $l^* \dashv k_*$.

A 2-categorical version of this result can is given in [4, Proposition 4.26].

6.6 Frame reconstruction

In [7] Macnab considered the problem of reconstructing a frame H from a given finite family of frame quotients of H. To achieve this reconstruction the family has to be jointwise monic, and the reconstruction used is a generalized version of the Artin construction. Thus we work in $\mathsf{Slt}^\wedge$ which is finitely enriched in the downward orientation. In particular, each finite web has an essentially unique spider. Now let H be any $\mathsf{Slt}^\wedge$-object and let

$$H \xrightarrow{\phi^i \dashv \phi_i} A(i)$$

$(1 \leq i \leq m)$ be a finite family of adjoint pairs in $\mathsf{Slt}^\wedge$. (In particular, if H is a frame then such a family is given by any family of frame quotients of H.) For each $1 \leq i, j \leq m$ let

$$A(j, i) \ = \ \phi^j \phi_i.$$

We easily check that this gives a down web $\mathfrak{A}$ with nodes $A(1), \ldots, A(m)$. By construction the family

$$H \xrightarrow{\phi^\bullet \dashv \phi_\bullet} \mathfrak{A}$$

provides both a cone and a co-cone for $\mathfrak{A}$. Furthermore, this family structures H as a spider exactly when

$$\phi_1 \phi^1 \wedge \cdots \wedge \phi_m \phi^m \ = \ 1_H$$

and in this case $\mathcal{G}(\mathfrak{A})$ is canonically isomorphic to H.

6.7 Sierpinski cover

Suppose the category Cat is finitely enriched and consider any Cat-object A. There are two obvious members of $\mathsf{Cat}[A, A]$ namely 1_A and the extreme arrow. We may pair these to produce an Artin web and, after glueing, obtain a new object $Sier(A)$. From the study of a particular case I suggest that this assignment

$$A \longmapsto Sier(A)$$

has some hidden delights. Let me spell out some of the details of a particular case.

Consider the case of the category $\mathsf{Slt}^\wedge$ and suppose that the object A is a distributive lattice (not just a $\wedge-$semilattice). The web then has arrows

$$\begin{tikzcd} A \arrow[r, shift left, "1_A"] & A. \arrow[l, shift left, "\top"] \end{tikzcd}$$

Let

$$\begin{tikzcd} A \arrow[r, shift right, "f_*"'] & Sier(A) \arrow[l, shift right, "f^*"'] \arrow[r, shift left, "g^*"] & A \arrow[l, shift left, "g_*"] \end{tikzcd}$$

be the glued version of this web. We know that $Sier(A)$ consists of all pairs (a,b) of elements of A with $a \geq b$. Furthermore, $Sier(A)$ is a distributive lattice under the pointwise operations. The structuring arrows are given by

$$f^*(a,b) = a \quad , \quad g^*(a,b) = b$$

$$f_*(a) = (a,a) \quad , \quad g_*(b) = (\top, b).$$

Note that all of these are lattice morphisms (not just semilattice morphisms) and, in fact, the map f_* is a lattice embedding. We write

$$A \xrightarrow{\ \iota\ } Sier(A)$$

for this embedding and, more often than not, identify $a \in A$ with its image $\iota(a) \in Sier(A)$. We develop a more suggestive notation for elements of $Sier(A)$.

Let X be the element $(\top, \bot)$. Consider any other element $(a,b) \in Sier(A)$. Remembering that $a \geq b$, a simple calculation gives

$$(a,b) \ = \ (a,a) \wedge (\top, \bot) \vee (b,b) \ = \ a \wedge X \vee b$$

(where the expression doesn't need bracketting since $Sier(A)$ is distributive). This gives us a canonical representation of the elements p of $Sier(A)$. We think of these p as polynomials in the variable X with coefficients a and b. Using this notation the three other mediating maps are given by

$$f^*(p) = p[X := \top], \quad g^*(p) = p[X := \bot]$$

$$g_*(b) = X \vee b$$

for each $p \in Sier(A)$ and $b \in A$, where the two upper maps are evaluation at $\top$ and $\bot$, respectively.

Consider now any distributive lattice L and a lattice morphism

$$A \xrightarrow{\ \phi\ } L.$$

Consider any element $\ell \in L$. This gives us a co-wedge in Slt

$$\begin{tikzcd} & L & \\ A \arrow[ur, "\phi_*"] & & A \arrow[ul, "\psi_*"'] \end{tikzcd}$$

where

$$\phi_* = \phi \quad , \quad \psi_*(a) = \ell \vee \phi(a)$$

(for $a \in A$). Then

$$\psi_* 1(a) = \psi_*(a) = \ell \vee \phi(a) \geq \phi_*(a)$$

$$\psi_*\top(a) = \psi_*(\top) = \top \geq \psi_*(a)$$

for each $a \in A$, so that the co-wedge is a co-cone for the parent web.

On general grounds this gives us a unique **Slt**-arrow

$$Sier(A) \xrightarrow{\phi_\flat} L$$

such that

$$\phi_* = \phi_\flat f_* \quad , \quad \psi_* = \phi_\flat g_*$$

and in particular

$$\phi_\flat(a) = \phi(a) \quad , \quad \phi_\flat(X) = \ell$$

(for each $a \in A$). Finally, remembering the construction of $\phi_\flat$ as $\phi_* f^* \wedge \psi_* g^*$, we see that for each polynomial

$$p = a \wedge X \vee b$$

we have

$$\phi_\flat(p) \ = \ \phi_*(a) \wedge \psi_*(b) \ = \ \phi(a) \wedge \ell \vee \phi(b) \ = \ p[X; = \ell].$$

In particular $\phi_\flat$ is also a lattice morphism.

These calculations prove the following.

THEOREM. *For each lattice morphism*

$$A \xrightarrow{\phi} L$$

to a distributive lattice L and each element $\ell \in L$, there is a unique lattice morphism

$$Sier(A) \xrightarrow{\phi_\flat} L$$

such that

$$\begin{tikzcd} A \arrow[rr, "\phi"] \arrow[dr, "\iota"'] & & L \\ & Sier(A) \arrow[ur, "\phi_\flat"'] & \end{tikzcd}$$

commutes. That is, $Sier(A)$ is the distributive lattice freely generated over A by the addition of one new element.

Notice that in the statement of this result there appears to be a confusion between the left (free) object $Sier(A)$ and right attributes $\phi_\flat$. I believe this is significant.

Starting from the 2-element lattice $\mathbf{2}$ let $L_0, L_1, L_2, \ldots$ be the sequence of lattices defined by

$$L_0 = \mathbf{2} \quad , \quad L_{n+1} = Sier(L_n).$$

Then the Theorem shows that for each $n \leq \omega$, the object L_n is the free distributive lattice on n generators. I originally discovered the Theorem by explicitly going through the glueing construction to produce L_1, L_2 and L_3, and recognizing the resulting lattices for what they are. A version of this result appears in [5, p 32].

6.8 Biproducts

Consider any I-indexed family

$$\mathcal{A} \;=\; (A(i) \mid i \in I)$$

of objects and suppose that Cat is up I-nice. In particular, between any two objects there is a bottom arrow $\perp$, and for $i, j \in I$ we may set

$$A(j,i) \;=\; \begin{cases} 1_{A(i)} & \text{if } i = j \\ \perp & \text{if } i \neq j \end{cases}$$

to obtain an up web $\mathfrak{A}$ with nodes $\mathcal{A}$. A lax cone for $\mathfrak{A}$ is precisely a wedge

$$B \xrightarrow[\phi_i]{} A(i)$$

($i \in I$) of arrows (with no required compatibility conditions). Similarly, a lax co-cone is a co-wedge

$$B \xleftarrow{\phi^i} A(i)$$

($i \in I$) of arrows.

Now consider any spider

$$A \xleftarrow{f^\bullet \dashv f_\bullet} \mathfrak{A}$$

of $\mathfrak{A}$. Given any wedge

$$B \xleftarrow[\phi_\bullet]{} \mathfrak{A}$$

we may set

$$\phi_* \;=\; \bigvee\{f^i\phi_i \mid i \in I\}$$

to obtain a mediating arrow

$$B \xrightarrow[\phi_*]{} A$$

with $\phi_\bullet \;=\; f_\bullet\phi_*$. Similarly, for any co-wedge

$$B \xrightarrow{\phi^\bullet} \mathfrak{A}$$

we may set

$$\phi^* \;=\; \bigvee\{\phi^i f^i \mid i \in I\}$$

to obtain a mediating arrow

$$B \xleftarrow{\phi^*} A$$

with $\phi^\bullet \;=\; \phi^* f^\bullet$. This (with a couple more observations) shows that the spider is the biproduct of the family $\mathcal{A}$.

A similar result holds if Cat is down I-nice.

In particular, this shows that $\mathsf{Slt}^\vee$ and $\mathsf{Slt}^\wedge$ have biproducts of all finite families, and Sup and Inf have biproducts of all families. In these concrete cases a thread of $\mathfrak{A}$ is precisely a choice function of $\mathcal{A}$. The details of these examples are worth going through. Many of the calculations with general webs and spiders are mild generalizations of these particular cases.

6.9 Initial and final sections

For a given poset $(S, \sqsubseteq)$ let

$$\Lambda S \quad | \quad \Upsilon S \tag{3}$$

be, respectively, the lattice of

$$\text{lower} \quad | \quad \text{upper}$$

sections of S. I show how these occur naturally as the glued versions of certain

$$\text{up} \quad | \quad \text{down}$$

webs $\mathfrak{S}$ indexed by S. As usual I will describe the two versions in parallel, and for this it is convenient to write

$$\Sigma S$$

for the particular version of (3) which is relevant to the current orientation.

We need to look at the 2-element lattice

$$\mathsf{2} = \left\{ \begin{matrix} 1 \\ \\ 0 \end{matrix} \right\}$$

which lives in many different categories and schizophrenically induces many important contravariant adjunctions. As a poset $\mathsf{2}$ carries three endomorphisms

$$\bot \quad , \quad \mathsf{1} \quad , \quad \top$$

where $\mathsf{1}$ is the identity map on $\mathsf{2}$ and the other two are determined by

$$\bot(1) = 0 \quad | \quad \top(0) = 1.$$

Of these, only

$$\bot\,,\, 1 \quad | \quad 1\,,\,\top$$

are

$$\mathsf{Slt}^\vee \quad | \quad \mathsf{Slt}^\wedge$$

arrows.

Now take $I = S$ as the index set and for each $p \in S$ let

$$A(p) = \mathsf{2}.$$

For each $p, q \in S$ let

$$A(q,p) = \begin{cases} \mathsf{1} & \text{if } q \sqsubseteq p \\ \bot & \text{if } q \not\sqsubseteq p \end{cases} \quad \Bigg| \quad A(q,p) = \begin{cases} \top & \text{if } q \not\sqsubseteq p \\ \mathsf{1} & \text{if } q \sqsubseteq p \end{cases}$$

so that the crucial value is

$$A(q,p)(1) = \begin{cases} 1 & \text{if } q \sqsubseteq p \\ 0 & \text{if } q \not\sqsubseteq p. \end{cases} \quad \Bigg| \quad A(q,p)(0) = \begin{cases} 1 & \text{if } q \not\sqsubseteq p \\ 0 & \text{if } q \sqsubseteq p. \end{cases}$$

A simple calculation shows that this defines an S-indexed

up | down

web $\mathfrak{S}$ in

$$\mathsf{Slt}^{\vee} \quad | \quad \mathsf{Slt}^{\wedge}.$$

You should check that the threads of $\mathfrak{S}$ are those functions

$$a : S \longrightarrow \mathsf{2}$$

which are

antitone | monotone.

These are in bijective correspondence with the members U of ΣS where the correspondence

$$a \longleftrightarrow U$$

is given by

$$a(p) = 1 \;\Leftrightarrow\; p \in U$$

(for $p \in S$).

For each $p \in S$ let

$$\mathsf{2} \xrightarrow{f^p \dashv f_p} \Sigma S \quad | \quad \Sigma S \xrightarrow{f^p \dashv f_p} \mathsf{2}$$

be the canonical adjoint pair for p given by

$$\begin{array}{lcl|lcl} f^p(1) & = & {\downarrow}p & f^p(U) & = & 1 \Leftrightarrow p \in U \\ f_p(U) & = & 1 \Leftrightarrow p \in U & f_p(0) & = & ({\downarrow}p)' \end{array}$$

(for $U \in \Sigma S$). You should check that

$$\bigvee\{f^p f_p \mid p \in S\} \;=\; 1_{\Sigma S} \quad\Big|\quad \bigwedge\{f_p f^p \mid p \in S\} \;=\; 1_{\Sigma S}$$

and, in fact, the family

$$2 \xrightarrow{f^\bullet \dashv f_\bullet} \Sigma S \quad\Big|\quad \Sigma S \xrightarrow{f^\bullet \dashv f_\bullet} 2$$

structures ΣS as the glued version of $\mathfrak{S}$.

This web $\mathfrak{S}$ has many different spiders. To see this let H be any

$$\bigvee\text{-semilattice} \quad\Big|\quad \text{frame}$$

and take

$$S \;=\; H \quad\Big|\quad S \;=\; pt(H)$$

where $pt(H)$ is the set of points of H in the guise of the $\wedge$-irreducible elements. The ordering on $pt(H)$ is the *converse* of the one on H. We have a structuring adjoint pair

$$\Sigma S \xrightarrow{h^* \dashv h_*} H \quad\Big|\quad H \xrightarrow{h^* \dashv h_*} \Sigma S$$

given by

$$h^*(U) \;=\; \bigvee U \quad\Big|\quad h^*(u) \;=\; \{p \in S \mid u \not\leq p\}$$

$$h_*(u) \;=\; {\downarrow} u \quad\Big|\quad h_*(U) \;=\; \bigwedge(S - U)$$

for each $u \in H$ and $U \in \Sigma S$, where the infimum of $h_*(U)$ is computed in H (not S).

At this stage we discover a difference between the two cases since

$$h^* h_* \;=\; 1_H \quad\Big|\quad h_* h^* \;=\; 1_H \;\Leftrightarrow\; H \text{ is spatial.}$$

To preserve the symmetry let us assume that, in the down case, the frame H is spatial.

For each $p \in S$ let

$$2 \xrightarrow{h^p \dashv h_p} H \quad\Big|\quad H \xrightarrow{h^p \dashv h_p} 2 \tag{4}$$

be the composite

$$2 \xrightarrow{f^p \dashv f_p} \Sigma S \xrightarrow{h^* \dashv h_*} H \quad\Big|\quad H \xrightarrow{h^* \dashv h_*} \Sigma S \xrightarrow{f^p \dashv f_p} 2.$$

It is easy to check that for each $p, q \in S$ we have

$$h_q h^p \;=\; A(q,p) \quad\Big|\quad h^q h_p \;=\; A(q,p)$$

so that the family $h^\bullet \dashv h_\bullet$ of adjoint pairs structures H as a a cone and a co-cone for $\mathfrak{S}$. Also, remembering the preservation properties of adjoint pairs, we find that

$$\begin{array}{l|l} \bigvee\{h^p h_p \mid p \in S\} & \bigwedge\{h_p h^p \mid p \in S\} \\ \quad = h^*(\bigvee\{f^p f_p \mid p \in S\}) h_* & \quad = h_*(\bigwedge\{f_p f^p \mid p \in S\}) h^* \\ \qquad\qquad = h^* h_* \;=\; 1_H & \qquad\qquad = h_* h^* \;=\; 1_H. \end{array}$$

Thus we see that the family (4) structures H as a spider for $\mathfrak{S}$. Since the mediating pair $h^* \dashv h_*$ need not be an isomorphism this shows that a web can have more than one (in fact infinitely many) distinct spiders.

7 Filtered co-limits

Almost all the applications of glueing which have so far been recognized as such take place in the down orientation. An important example of an up application is the Smyth-Plotkin analysis [9] of the construction of certain bilimits. I will explain this in some detail.

Let Cat be an enriched category whose arrow sets are at least up directedly complete. Thus, for each directed set G of arrows

$$B \xrightarrow{\quad g \quad} C$$

(between a common source and target) the supremum $\bigvee G$ exists. Furthermore, for all arrows

$$A \xrightarrow{\quad f \quad} B \qquad , \qquad C \xrightarrow{\quad h \quad} D$$

we have

$$h(\bigvee G)f \;=\; \bigvee\{hgf \mid g \in G\}.$$

In short, modulo the distinction between directed sets and ω-chains, Cat is an O-category in the sense of Smyth and Plotkin [9] and Wand [10]. The arrows of $\mathsf{Cat}^{\dashv}$ are, of course, the adjoint pairs

$$\begin{tikzcd} A \arrow[r, shift left, "f^*"] & B \arrow[l, shift left, "f_*"] \end{tikzcd}$$

of Cat-arrows satisfying

$$1_A \leq f_* f^* \quad , \quad f^* f_* \leq 1_B.$$

Let I be any directed poset based on the set I. We use this to index a (strict) diagram $\mathbb{A}^{\dashv}$ in $\mathsf{Cat}^{\dashv}$. This is given by a functor

$$I \xrightarrow{\qquad} \mathsf{Cat}^{\dashv}$$

so that, in detail, we have:

- for each $i \in I$, an object $A(i)$;
- for each $i, j \in I$ with $i \leq j$, a $\mathsf{Cat}^{\dashv}$-arrow

$$A(i) \xrightarrow{\quad A^*(j,i) \dashv A_*(i,j) \quad} A(j)$$

with the appropriate commuting properties, i.e. such that

$$A^*(k,j)A^*(j,i) = A^*(k,i) \quad , \quad A_*(i,j)A_*(j,k) = A_*(i,k)$$

for all $i \leq j \leq k$.

By taking the left and right components of the arrows of $\mathbb{A}^{\dashv}$ we obtain two strict diagrams

$$\mathbb{A}^{*} \quad , \quad \mathbb{A}_{*}$$

in Cat, both given by functors

$$I \longrightarrow \mathsf{Cat}$$

with the first covariant and the second contravariant. We are interested in

- the co-limit (in $\mathsf{Cat}^{\dashv}$) of $\mathbb{A}^{\dashv}$,
- the co-limit (in Cat) of $\mathbb{A}^{*}$,
- the limit (in Cat) of $\mathbb{A}_{*}$,

in particular we wish to show that these gadgets are one and the same.

To obtain these universal objects we take the (unique) spider of a certain web $\mathfrak{A}$ which, in turn, is obtained from $\mathbb{A}^{\dashv}$. Thus our first job is to describe $\mathfrak{A}$.

The web $\mathfrak{A}$ is I-indexed, its nodes are just the nodes of $\mathbb{A}^{\dashv}$, and its arrows are certain combinations of the components of the adjoint pairs of $\mathbb{A}^{\dashv}$. Since we are concerned with computing with up directed sets we use the up version of the general glueing technique. Thus we must produce Cat-arrows

$$A(s) \xrightarrow{A(t,s)} A(t)$$

such that

- $1_A(r) \le A(r,r)$
- $A(t,s)A(s,r) \le A(t,r)$

for all $r, s, t \in I$.

Before we define the arrows $A(t,s)$ we make a preliminary observation.

7.1 LEMMA. *(a) For all $i \le s, t \le j$ in I and associated square*

$$\begin{tikzcd}
 & A(s) \arrow[dl, "A_*(i,s)"'] \arrow[dr, "A^*(j,s)"] & \\
A(i) \arrow[dr, "A^*(t,i)"'] & \le & A(j) \arrow[dl, "A_*(t,j)"] \\
 & A(t) &
\end{tikzcd}$$

the comparison

$$A^*(t,i)A_*(i,s) \le A_*(t,j)A^*(j,s)$$

holds.

(b) For all $s, t \leq i \leq j$ in I and associated diagram

$$\begin{tikzcd}
A(s) \arrow[dr, "A^*(i,s)"'] \arrow[drr, "A^*(j,s)"] & & \\
& A(i) \arrow[dl, "A_*(t,i)"'] \arrow[r, phantom, "\leq"] & A(j) \arrow[dll, "A_*(j,t)"] \\
A(t) & &
\end{tikzcd}$$

the comparison

$$A_*(t,i)A^*(i,s) \leq A_*(t,j)A^*(j,s)$$

holds.

Proof. (a) Let

$$LHS \;=\; A^*(t,i)A_*(i,s) \quad , \quad A_*(t,j)A^*(j,s) \;=\; RHS.$$

Then, using the adjoint pair comparisons, we have

$$\begin{array}{rcll}
LHS & = & A^*(t,i)A_*(i,s)1_A(s) & \\
& \leq & A^*(t,i)A_*(i,s)A_*(s,j)A^*(j,s) & \\
& = & A^*(t,i)A_*(i,j)A^*(j,s) & \\
& = & A^*(t,i)A_*(i,t)A_*(t,j)A^*(j,s) & \\
& \leq & 1_A(t)RHS & = \quad RHS
\end{array}$$

as required.

(b) Let

$$LHS \;=\; A_*(t,i)A^*(i,s) \quad , \quad A_*(t,j)A^*(j,s) \;=\; RHS.$$

Then, using the adjoint pair comparisons, we have

$$\begin{array}{rcll}
LHS & = & A_*(t,i)1_A(i)A^*(i,s)1_A(s) & \\
& \leq & A_*(t,i)A_*(i,j)A^*(j,i)A^*(i,s) & \\
& = & A_*(t,j)A_*(j,s) & = \quad RHS
\end{array}$$

as required. ■

For $s, t \in I$ let $\uparrow(s,t)$ be the set of all $i \in I$ with $s, t \leq i$. Part (b) of Lemma 7.1 shows that

$$\Delta(t,s) \;=\; \{A_*(t,i)A^*(i,s) \mid i \in \uparrow(t,s)\}$$

is a directed subset of $\mathsf{Cat}[A(s), A(t)]$ which, with the assumed condition on Cat, allows the following.

7.2 **DEFINITION**. For each $s, t \in I$ let

$$A(t,s) \;=\; \bigvee \Delta(t,s)$$

where $\Delta(t,s)$ is the directed set given above. ∎

Given any $s, t \in I$ there are many different composite arrows from $A(s)$ to $A(t)$ formed by zig-zagging through the adjoint pairs of $\mathbb{A}^{\dashv}$. You may like to check that such an arrow is always bounded above by $A(t,s)$. You should also note that if the arrows of $\mathbb{A}^{\dashv}$ form projection pairs, then all members of $\Delta(t,s)$ are equal.

7.3 **LEMMA**. *The assignments*

$$i \longmapsto A(i) \qquad , \qquad (j,i) \longmapsto A(j,i)$$

form an I-indexed web in Cat.

Proof. Firstly, for each $r \in I$ we have

$$A(r,r) \;\geq\; A_*(r,r)A^*(r,r) \;\geq\; 1_A(r).$$

For the second comparison consider any $r, s, t \in I$, take any $i, j \in I$ with $r, s \leq i$ and $s, t \leq j$, and then take any $k \in I$ with $i, j \leq k$. This gives us a diagram

$$\begin{tikzcd}
A(r) \arrow[dr, "A^*(t,s)"] & & \\
& A(i) \arrow[dl, "A_*(i,s)"'] \arrow[dr, "A^*(k,i)"] & \\
A(s) \arrow[dr, "A^*(j,s)"'] & & A(k) \arrow[dl, "A^*(j,k)"] \\
& A(j) \arrow[dl, "A_*(j,t)"] & \\
A(t) & &
\end{tikzcd}$$

in Cat. We apply part (a) of Lemma 7.1 to the square of this diagram to get

$$\begin{array}{rcll}
A(t,s)A(s,r) & = & \bigvee\{A_*(t,j)A^*(j,s)A_*(s,i)A^*(i,r) \mid j \in\uparrow(t,s), i \in\uparrow(s,r)\} & \\
& \leq & \bigvee\{A_*(t,j)A_*(j,k)A^*(k,i)A^*(i,r) \mid j \in\uparrow(t,s), i \in\uparrow(s,r), k \in\uparrow(j,i)\} & \\
& = & \bigvee\{A_*(t,k)A^*(k,t) \mid j \in\uparrow(t,s), i \in\uparrow(s,r), k \in\uparrow(j,i)\} & \leq \;\; A(t,r)
\end{array}$$

as required. ∎

Let $\mathfrak{A}$ be this web. We will relate the spiders of $\mathfrak{A}$ to the required universal gadgets of $\mathbb{A}^{\dashv}, \mathbb{A}^*$, and $\mathbb{A}_*$. To do this we use the following observation.

7.4 LEMMA. $(\cdot)^*$. *For each co-wedge*

$$A(i) \xrightarrow{\phi^i} B$$

in Cat *(for $i \in I$), the two properties:*

- *The co-wedge is a lax co-cone for $\mathfrak{A}$.*
- *The co-wedge is a strict co-cone for $\mathbb{A}^*$.*

are equivalent.

$(\cdot)_*$. *For each wedge*

$$A(i) \xleftarrow[\phi_i]{} B$$

in Cat *(for $i \in I$), the two properties:*

- *The wedge is a lax cone for $\mathfrak{A}$.*
- *The wedge is a strict cone for $\mathbb{A}_*$.*

are equivalent.

Proof. $(\cdot)^*$. Suppose first that the given co-wedge is a lax co-cone for $\mathfrak{A}$. Thus we have

$$\phi^s \;\geq\; \phi^t A(t,s)$$

for all $s, t \in I$. In particular, for $i \leq j$ we have

$$\phi^i \;\geq\; \phi^j A^*(j,i) \qquad \text{and} \qquad \phi^j \;\geq\; \phi^i A_*(i,j).$$

The second of these gives

$$\phi^j A^*(j,i) \;\geq\; \phi^i A_*(i,j) A^*(j,i) \;\geq\; \phi^i$$

so that $\phi^i \;=\; \phi^j A^*(j,i)$ as required to show that the given co-wedge is a strict co-cone for $\mathbb{A}^*$.

Conversely suppose that we know the given co-wedge is a strict co-cone for $\mathbb{A}^*$. For arbitrary $s, t \in I$ take any $i \in \uparrow(s,t)$ so that, since

$$\phi^t \;=\; \phi^i A^*(i,t),$$

we have

$$\phi^t A_*(t,i) \;=\; \phi^i A^*(i,t) A_*(t,i) \; leq \; \phi^i$$

and hence

$$\phi^t A(t,s) \;=\; \bigvee\{\phi^t A_*(t,i) A^*(i,s) \mid i \in \uparrow(t,s)\} \;\leq\; \bigvee\{\phi^i A^*(i,s) \mid i \in \uparrow(t,s)\} \;\leq\; \phi^s$$

as required to show that the given co-wedge is a lax co-cone for $\mathfrak{A}$.

$(\cdot)_*$. This follows by a dual argument ■

7.5 LEMMA. *For all $i, j, s, t \in I$ with $s \le i$ and $t \le j$, the equality*

$$A_*(t,j)A(j,i)A^*(i,s) \;=\; A(t,s)$$

holds.

Proof. The compound arrow LHS on the left hand side is given by

$$\begin{aligned} LHS &= \bigvee\{A_*(t,j)A_*(j,k)A^*(k,i)A^*(i,s) \mid k \in\uparrow(j,i)\} \\ &= \bigvee\{A_*(t,k)A^*(k,s) \mid k \in\uparrow(j,i)\} \qquad = \quad A(t,s) \end{aligned}$$

(using part (b) of Lemma 7.1), as required ■

With these preliminaries in place we can now obtain the main results.
Let A be any spider for $\mathfrak{A}$ with structuring arrows

$$\begin{tikzcd} A(i) \arrow[r, shift left, "f^i"] & A. \arrow[l, shift left, "f_i"] \end{tikzcd}$$

As usual, these satisfy

$$f_t f^s \;=\; A(t,s) \quad , \quad \bigvee\{f^i f_i \mid i \in I\} \;=\; 1_A$$

$$f^s \ge f^t A(t,s) \quad , \quad f_t \ge A(t,s) f_s$$

(for $s, t \in I$).

7.6 THEOREM. $(\cdot)^*$. *The arrows*

$$A(i) \xrightarrow{\;f^i\;} A$$

structure A as the co-limit of $\mathbb{A}^*$.

$(\cdot)_*$. *The arrows*

$$A(i) \xleftarrow[\;f_i\;]{} A$$

structure A as the limit of $\mathbb{A}_*$.

$(\cdot)^\dashv$. *The adjoint pairs*

$$A(i) \xrightarrow{\;f^i \dashv f_i\;} A$$

structure A as the co-limit of $\mathbb{A}^\dashv$.

Proof. $(\cdot)^*$. The arrows f^i form a lax co-cone for $\mathfrak{A}$ and hence, by Lemma 7.4, form a strict co-cone for $\mathbb{A}^*$. Consider any other strict co-cone

$$A(i) \xrightarrow{\phi^i} B$$

for $\mathbb{A}^*$. Again using Lemma 7.4, we see that these arrows form a lax co-cone for $\mathfrak{A}$. We must produce an appropriate mediating arrow

$$A \xrightarrow{\phi^*} B.$$

For each $i \in I$, consider the arrow

$$A \xrightarrow{\phi^i f_i} B.$$

For each $i \leq j$ we have

$$\phi^i f_i \;=\; \phi^j A^*(j,i) f_i \;\leq\; \phi^j A(j,i) f_i \;\leq\; \phi^j f_j$$

so we may set

$$\phi^* \;=\; \bigvee\{\phi^i f_i \mid i \in I\}.$$

With this arrow we have

$$\phi^* f^s \;=\; \bigvee\{\phi^i f_i f^i \mid i \in I\} \;=\; \bigvee\{\phi^i A(i,s) \mid i \in I\} \;\leq\; \phi^s$$

and

$$\phi^* f^s \;\geq\; \phi^s A(s,s) \;\geq\; \phi^s$$

so that $\phi^* f^s \;=\; \phi^s$ to show that ϕ^* is a mediating arrow.

Finally, for any arrow ϕ with $\phi f^i \;=\; \phi^i$ for all $i \in I$, we have

$$\phi \;=\; \phi \mathbf{1}_A \;=\; \bigvee\{\phi f^i f_i \mid i \in I\} \;=\; \bigvee\{\phi^i f_i \mid i \in I\} \;=\; \phi^*$$

to prove the required uniqueness.

$(\cdot)_*$. This follows by a dual argument.

$(\cdot)^{\dashv}$. We already know that the pairs

$$A(i) \xrightarrow{f^i \dashv f_i} A$$

form a co-cone for $\mathbb{A}^{\dashv}$. Furthermore, given any co-cone

$$A(i) \xrightarrow{\phi^i \dashv \phi_i} B$$

for $\mathbb{A}^{\dashv}$, we know there are unique arrows

$$\begin{tikzcd} A \arrow[r, shift left, "\phi^*"] & B \arrow[l, shift left, "\phi_*"] \end{tikzcd}$$

with

$$\phi^i \;=\; \phi^* f^i \quad , \quad \phi_i \;=\; f_i \phi_*$$

for all $i \in I$. It suffices to show that $\phi^* \dashv \phi_*$.

Recalling the definitions of ϕ^* and ϕ_*, for each $i \in I$ we have

$$\phi_*\phi^* \;\geq\; f_i\phi^i\phi_i f^i \;\geq\; f_i 1_B f^i \;=\; f_i f^i$$

so that

$$\phi_*\phi^* \;\geq\; \bigvee\{f_i f^i \mid i \in I\} \;=\; 1_A.$$

Similarly

$$\begin{aligned} \phi^*\phi_* &= \bigvee\{\phi^j f_j f^i \phi_j \mid i,j \in I\} \\ &= \bigvee\{\phi^j A(j,i)\phi_j \mid i,j \in I\} \\ &\leq \bigvee\{\phi^i\phi_i \mid i,j \in I\} \qquad \leq \; 1_B \end{aligned}$$

as required to complete the proof. ■

Notice that one consequence of this result is that any two spiders for $\mathfrak{A}$ are canonically isomorphic.

The following result is the direct analogue of [9, Theorem 2]. You may like to provide a proof.

7.7 THEOREM. $(\cdot)^*$. *Suppose the arrows*

$$A(i) \xrightarrow{\;f^i\;} A$$

structure the object A as the co-limit of $\mathbb{A}^*$. Then these arrows form the left components of adjoints pairs which structure A as a spider for $\mathfrak{A}$.

$(\cdot)_*$. *Suppose the arrows*

$$A(i) \xleftarrow[\;f_i\;]{} A$$

structure the object A as the limit of $\mathbb{A}^*$. Then these arrows form the right components of adjoints pairs which structure A as a spider for $\mathfrak{A}$.

$(\cdot)^{\dashv}$. *Suppose the arrows*

$$A(i) \xrightarrow{\;f^i \dashv f_i\;} A$$

structure the object A as the co-limit of $\mathbb{A}^{\dashv}$. Then these arrows structure A as a spider for $\mathfrak{A}$.

The construction of this section is also related to the result of Isbell given in [3] concerning the coincidence of direct and inverse limits.

8 Concluding remarks

For a suitable category Cat I have set up a category Catl of lax diagrams over Cat together with a canonical functor

$$\mathsf{Cat} \xrightarrow{\mathcal{F}} \mathsf{Catl}.$$

We have seen that globally the glueing construction provides a right adjoint to $\mathcal{F}$ and locally it provides a limit for the diagrams of Catl. From a categorical point of view there are two obvious questions.

(1) When does $\mathcal{F}$ have a left adjoint?

(2) When does a diagram $\mathfrak{A}$ of Catl have a co-limit?

Observe that a co-limit for $\mathfrak{A}$ will be a certain co-cone for $\mathfrak{A}$. Each spider for $\mathfrak{A}$ produces such a co-cone. Thus it could be that, as with limits, the required co-limit for $\mathfrak{A}$ can be found amongst these spiders.

(3) Suppose $\mathfrak{A}$ has a unique spider. Is this spider both a limit and a co-limit for $\mathfrak{A}$?

For this question another look at Theorem 4.8 might provide some helpful information.

All the diagrams (webs) of this paper are indexed by an indiscrete category. There are other kinds of diagrams to be considered.

(4) What happens if we consider (lax) diagram indexed by a more finely structured category?

We could, of course, use an indexing category which is itself enriched with some laxness. This has the possibility of increasing the complexity of the analysis no end. Thus to answer this question some guiding examples should be followed.

(5) What happens if mixed laxness is allowed (even over an indiscrete indexing category)?

There are places in the present analysis which suggest that such a mixed approach might be worthwhile.

These five questions are concerned with a deeper analysis of the algebraic version of glueing. There is a different, and perhaps more significant, direction.

(6) How can the analysis of this paper be lifted to a 2-categorical setting?

For this question the paper [11] by Wraith and the account [4, pp 107 – 112] should be the starting point.

All of the significant applications of glueing to proof theory are done in a 2-categorical setting, for instance see [6, sections 20 & 21]. In these applications a topos $\mathcal{T}$ is glued to a fairly simple category across a fringe functor using the analogue of the Artin construction.

(7) Are there any proof theoretic applications of glueing a non-Artin diagram?

After glueing, all of the proof theoretic applications proceed by an analysis of the internal structure of the glued gadget. The whole purpose of this paper is a first step towards answering the following.

> (8) How does the universal properties of the glued gadget control the proof theoretic applications?

I am, however, still completely baffled by this question.

References

[1] S. Abramsky and S. Vickers: Quantales, observational logic, and process semantics, Imperial College Research Report DOC 90/1.

[2] A. Grothendieck and J.L. Verdier: "Théories des Topos" (SGA 4, exposés I – VI). Second edition, Springer Lecture Notes in Mathematics 269 - 270 (1972).

[3] J. R. Isbell: Direct limits of meet-continuous lattices, J. Pure Appl. Algebra 23 (1982) 33 - 35.

[4] P. T. Johnstone: Topos theory, Academic Press (1977).

[5] P. T. Johnstone: Stone spaces, Cambridge U. P. (1982).

[6] J. Lambek and P. J. Scott: Introduction to higher order categorical logic, Cambridge U. P. (1986).

[7] D. S. Macnab: An algebraic study of modal operators on heyting algebras with applications to topology and sheafification, Doctoral dissertation, University of Aberdeen 1976.

[8] R. Sikorski: Applications of topology to the foundations of mathematics. pp 322 – 330 of General topology and its Relation to Modern Analysis and Algebra (Proceedings of the 1961 Prague Symposium), Academic Press (1962).

[9] M. B. Smyth and G. D. Plotkin: The category-theoretic solution of recursive domain equations, SIAM J. Computing 11 (1983) 761 - 783.

[10] M. Wand: Fixed-point constructions in order-enriched categories, Theoret. Comput. Sci. 8 (1979) 13 - 30.

[11] G. C. Wraith: Artin glueing, J. Pure and Applied Algebra 4 (1974) 345 - 348.

[12] J. B. Wright, E. G. Wagner, and J. W. Thatcher: A uniform approach to inductive posets and inductive closure, Theoret. Comput. Sci. 8 (1978) 57 - 77.