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Abstract

We present sound, (weakly) complete and cut-free tableau systems for the propositional normal modal logics **S4.3**, **S4.3.1** and **S4.14**. When the modality $\Box$ is given a temporal interpretation, these logics respectively model time as a linear dense sequence of points; as a linear discrete sequence of points; and as a branching tree where each branch is a linear discrete sequence of points.

Although cut-free, the last two systems do not possess the subformula property. But for any given finite set of formulae X the "superformulae" involved are always bounded by a finite set of formulae X^*_L depending only on X and the logic **L**. Thus each system gives a nondeterministic decision procedure for the logic in question. The completeness proofs yield deterministic decision procedures for each logic because each proof is constructive.

Each tableau system has a cut-free sequent analogue proving that Gentzen's cut-elimination theorem holds for these latter systems. The techniques are due to Hintikka and Rautenberg.

1 Introduction

The Diodorean modal logics **S4.3** and **S4.3.1** have received much attention in the literature because of their interpretation as logics of dense and discrete *linear* time [1]. The logics **S4** and **S4.14** can be given interpretations as logics of dense and discrete *branching* time. Using a technique due to Hintikka and Rautenberg we obtain sound and complete cut-free tableau systems for each logic. In so doing, we have to forsake the subformula property but, nevertheless, the tableau calculi give nondeterministic decision procedures for determining theoremhood. Furthermore, each tableau completeness proof is constructive and gives a deterministic decision procedure for the logic concerned.

Each (cut-free) tableau system has a (cut-free) sequent analogue thereby proving that Gentzen's cut-elimination theorem holds for each system. The resulting tableau systems are of interest from a theorem proving perspective since they are directly implementable in Prolog using a technique due to Fitting [5].

1.1 Preliminaries

The logics **S4.3**, **S4.3.1** and **S4.14** are all normal extensions of **S4** and are axiomatised by taking the rule of necessitation and modus ponens as inference rules, and by taking the appropriate formulae from Figure 1 as axiom schemas. Their respective axiomatisations are: **S4** is $KT4$; **S4.3** is $KT43$; **S4.3.1** is $KT43Dum$; and **S4.14** is $KT4Zbr$.

As usual, a Kripke frame is a pair $\langle W, R\rangle$ where W is a non-empty set (of possible worlds) and R is a binary relation on W. A Kripke model is a triple $\langle W, R, V\rangle$ where V is a mapping from primitive propositions to sets of worlds. Informally, if $\langle W, R\rangle$ is a frame where R is transitive, then a cluster C is a maximal subset of W such that for all *distinct* worlds w and w' in C we have wRw' and $w'Rw$. A cluster is **degenerate** if it is a single irreflexive world, otherwise it is **nondegenerate**. A nondegenerate cluster is **proper** if it consists of two or more worlds. A nondegenerate cluster is **simple** if it consists of a single reflexive world. Note that in a nondegenerate cluster, R is reflexive, transitive *and symmetric*. For an introduction to Kripke frames, Kripke models and the notion of clusters see Goldblatt [6] or Hughes and Cresswell [8].

We write $\vdash_L A$ to denote that A is a theorem of logic $\mathbf{L}$. Given a model $\langle W, R, V\rangle$ we write $w \models A$ to mean that w assigns "true" to A under the valuation V. A formula A is **valid in a model** $\langle W, R, V\rangle$ iff it is true in every world. A formula A is **valid in a frame** $\langle W, R\rangle$, written as $\langle W, R\rangle \models A$, iff it is valid in all models based on that frame.

Given a class of frames $\mathcal{C}$, logic $\mathbf{L}$ is **sound** with respect to $\mathcal{C}$ if for all frames $\mathcal{F} \in \mathcal{C}$ and all formulae A we have $\vdash_L A$ implies $\mathcal{F} \models A$. Logic $\mathbf{L}$ is **complete** with respect to $\mathcal{C}$ if for all frames $\mathcal{F} \in \mathcal{C}$ and all formulae A we have $\mathcal{F} \models A$ implies $\vdash_L A$. A logic $\mathbf{L}$ is **characterised** by a class of frames $\mathcal{C}$ iff $\mathbf{L}$ is sound and complete with respect to $\mathcal{C}$. The logics we study are known to be characterised by the classes of *finite* frames ascribed to them in Figure 2. We therefore define a frame to be an **L-frame** if it meets the appropriate criteria from Figure 2.

A model $\langle W, R, V\rangle$ is an $\mathbf{L}$-model iff $\langle W, R\rangle$ is an $\mathbf{L}$-frame. A formula A is $\mathbf{L}$-valid iff it is true in every world of every $\mathbf{L}$-model. An $\mathbf{L}$-model $\langle W, R, V\rangle$ is an $\mathbf{L}$-model for a finite set X iff there exists some $w_0 \in W$ such that $\forall A \in X, w_0 \models A$. A set X is $\mathbf{L}$-satisfiable iff there is an $\mathbf{L}$-model for X.

Axiom Name	Axiom Schema	Alternative Names
K	$\Box(A \Rightarrow B) \Rightarrow (\Box A \Rightarrow \Box B)$	
T	$\Box A \Rightarrow A$	M [10]
4	$\Box A \Rightarrow \Box\Box A$	
3	$\Box(\Box A \Rightarrow B) \vee \Box(\Box B \Rightarrow A)$	H [3], H_0^+ [9],Lem [13]
Dum	$\Box(\Box(A \Rightarrow \Box A) \Rightarrow A) \Rightarrow (\Diamond\Box A \Rightarrow \Box A)$	Dum_1 [13],$M1$ [7], $M14$ [18]
Zbr	$\Box(\Box(A \Rightarrow \Box A) \Rightarrow A) \Rightarrow (\Box\Diamond\Box A \Rightarrow \Box A)$	

Figure 1: Axiom names and alternative names.

L	**L**-frame
S4	finite, transitive tree of finite nondegenerate clusters [10]
S4.3	finite, reflexive, transitive sequence of finite nondegenerate clusters [8]
S4.3.1	finite, reflexive, transitive sequence of finite nondegenerate clusters with no proper non-final clusters [8]
S4.14	finite, reflexive, transitive tree of finite nondegenerate clusters with no proper non-final clusters [18]

Figure 2: Definition of L-frames.

It can be shown that

$$\langle \mathcal{I}, \leq \rangle \models A \text{ iff } \vdash_{S4.3} A$$

where $\mathcal{I}$ is either the set of real numbers or the set of rational numbers and $\leq$ is the usual ordering on numbers [6, page57]. Consequently, between any two points there is always a third and **S4.3** is said to model linear dense time. It can be shown that

$$\langle \omega, \leq \rangle \models A \text{ iff } \vdash_{S4.3.1} A$$

where ω is the set of natural numbers [6]. Hence, between any two points there is always a finite number (possibly none) of other points and **S4.3.1** is said to model linear discrete time. The correspondence between $\langle \mathcal{I}, \leq \rangle$ and **S4.3**-frames, and between $\langle \omega, \leq \rangle$ and **S4.3.1**-frames can be obtained by bulldozing proper clusters and defining an appropriate mapping called a p-morphism [6] [8].

It can be shown that **S4** is also characterised by the class of all reflexive transitive (and possibly infinite) trees [8, page 120]. That is, by bulldozing each proper cluster of an **S4**-frame we can obtain an infinite dense sequence so that **S4** is the logic that models branching dense time. The axiomatic system **S4.14** is proposed by Zeman [18, page 249] as the temporal logic for branching discrete time. The name **S4.14** is due to Zeman.

Therefore, the logics **S4**, **S4.3**, **S4.3.1** and **S4.14** cover the four possible combinations of discreteness and density paired with linearity and branching.

2 Modal Tableau Systems

The most popular tableau formulation is due to Smullyan as expounded by Fitting [4]. Following Rautenberg [10, 11], we use a slightly different formulation where formulae are carried from one tableau node to its child because the direct correspondence between sequent systems and tableau systems is easier to see using this formulation. We use a denumerable set of primitive propositions $\mathcal{P}$ and a constant false proposition 0. To minimise the number of rules, we work with primitive notation, taking $\Box, \neg$ and $\wedge$ as primitives and defining all other connectives from these. All our tableau systems work with *finite* sets of formulae.

We use the following notational conventions:

p, q denote primitive (atomic) propositions from $\mathcal{P}$;

P, Q, Q_i and P_i denote (well formed) formulae;

X, Y, Z denote *finite* (possibly empty) sets of (well formed) formulae;

$X; Y$ stands for $X \cup Y$ and $X; P$ stands for $X \cup \{P\}$;

$\Box X$ stands for $\{\Box P \mid P \in X\}$;

$\neg\Box X$ stands for $\{\neg\Box P \mid P \in X\}$.

A **tableau rule** consists of a **numerator** above the line and a list of **denominators** (below the line). The denominators are separated by vertical bars. The numerator is a finite set of formulae and so is each denominator. We use the terms numerator and denominator rather than premiss and conclusion to avoid confusion with the sequent terminology. Each tableau rule is read downwards as "if the numerator is satisfiable, then so is one of the denominators". A tableau calculus $\mathcal{C}L$ is a finite collection of tableau rules identified with the set of its rule names.

Following Rautenberg [10], a $\mathcal{C}L$-**tableau** for X is a finite tree $\mathcal{T}$ with root X whose nodes carry finite formula sets stepwise constructed by the rules of $\mathcal{C}L$ according to:

- if a rule with n denominators is applied to a node then that node has n successors with the proviso that
- if a node E carries a set Y and Y has already appeared on the branch from the root to E then E is an end node of $\mathcal{T}$.

A tableau is **closed** if all its end nodes carry $\{0\}$. A set X is $\mathcal{C}L$-**consistent** if no closed $\mathcal{C}L$-tableau for X exists.

Figure 3 shows a common tableau system for **S4**. Figure 4 shows the rules we need for **S4.3**, **S4.3.1** and **S4.14**. Figure 5 shows the sequent analogues of each rule. The only structural rule is (θ) and, in particular, there are no contraction or cut rules as shown below:

$$\text{(contraction)}\ \frac{X; P; P}{X; P} \qquad\qquad \text{(cut)}\ \frac{X}{X; P \mid X; \neg P}$$

When formulated using sets rather than multisets, tableau systems include an implicit rule of contraction since the set $X; P; P$ is the same as the set $X; P$. We believe that contraction is eliminable from our systems provided the (T) rule contains a form of contraction on $\Box P$ where $\Box P$ is carried from the numerator into the denominator. We return to this point later.

$$(0)\ \frac{P; \neg P}{0}$$

$$(\wedge)\ \frac{X; P \wedge Q}{X; P; Q} \qquad (\vee)\ \frac{X; \neg(P \wedge Q)}{X; \neg P \ \mid\ X; \neg Q}$$

$$(\neg)\ \frac{X; \neg\neg P}{X; P}$$

$$(T)\ \frac{X; \Box P}{X; \Box P; P} \qquad (S4)\ \frac{\Box X; \neg\Box P}{\Box X; \neg P}$$

$$(\theta)\ \frac{X; Y}{X}$$

Figure 3: Tableau rules for $\mathcal{C}S4$

The cut rule encodes the law of the excluded middle but suffers the disadvantage that the new formulae P and $\neg P$ are totally arbitrary, bearing no relationship to the numerator X. The redundancy of the cut rule is therefore very desirable and can be proved in two ways. The first is to allow the cut rule and to show syntactically that whenever there is a closed $\mathcal{C}L$-tableau for X containing uses of the cut rule, there is another closed $\mathcal{C}L$-tableau for X containing no uses of the cut rule. This is the cut-elimination theorem of Gentzen. The alternative is to omit the cut rule from the beginning and to show that the cut-free tableau system $\mathcal{C}L$ is nevertheless sound and complete with respect to the semantics of the logc $\mathbf{L}$. We follow this latter route and formalise this argument as follows.

Let (ρ) be an arbitrary tableau rule with a numerator $\mathcal{N}$ and n denominators $\mathcal{D}_1, \mathcal{D}_2, \cdots, \mathcal{D}_n$:

$$(\rho)\ \frac{\mathcal{N}}{\mathcal{D}_1 \mid \mathcal{D}_2 \mid \cdots \mid \mathcal{D}_n}$$

Let $\mathcal{C}L\rho$ be the tableau system $\mathcal{C}L \cup \{(\rho)\}$. Then the rule (ρ) is said to be **admissible** in $\mathcal{C}L$ if: X is $\mathcal{C}L$-consistent iff X is $\mathcal{C}L\rho$-consistent.

Lemma 1 *If $\mathcal{C}L$ is sound and complete with respect to $\mathbf{L}$-frames and (ρ) is sound with respect to $\mathbf{L}$-frames then (ρ) is admissible in $\mathcal{C}L$.*

Proof: Since $\mathcal{C}L \subseteq \mathcal{C}L\rho$ we know that if X is $\mathcal{C}L\rho$-consistent then X is $\mathcal{C}L$-consistent. To prove the other direction suppose that $\mathcal{C}L$ is sound and complete with respect to $\mathbf{L}$-frames, that (ρ) is sound with respect to $\mathbf{L}$-frames, and that X is $\mathcal{C}L$-consistent. By soundness, (ρ) preserves $\mathbf{L}$-satisfiability and hence, by the completeness of $\mathcal{C}L$, the set X must be $\mathbf{L}$-satisfiable. If X is not $\mathcal{C}L\rho$-consistent then there is a closed $\mathcal{C}L\rho$-tableau for X which must utilise the rule (ρ) since this is the only difference between $\mathcal{C}L$ and $\mathcal{C}L\rho$. But, by the soundness of $\mathcal{C}L$ and (ρ), this implies that X must be $\mathbf{L}$-unsatisfiable; contradiction. •

$$(S4.14)\ \frac{\Box X; \neg\Box P}{\Box X; \Box\neg\Box P \quad | \quad \Box X; \neg P; \Box(P \Rightarrow \Box P)}$$

$$(S4.3)\quad \frac{\Box X; \neg\Box\{P_1, \cdots, P_k\}}{\Box X; \neg\Box\overline{Y_1}; \neg P_1 \quad | \quad \cdots \quad | \quad \Box X; \neg\Box\overline{Y_k}; \neg P_k}$$

where $Y = \{P_1, \cdots, P_k\}$ and $\overline{Y_i} = Y \setminus \{P_i\}$

$$(S4.3.1)\quad \frac{U; \Box X; \neg\Box\{Q_1, \cdots, Q_k\}}{S_1 \quad | S_2 | \quad \cdots \quad | \quad S_k \quad | S_{k+1} | \quad S_{k+2} \quad | \quad \cdots \quad | \quad S_{2k}}$$

where

$Y = \{Q_1, \cdots, Q_k\};$

$\overline{Y_j} = Y \setminus \{Q_j\};$

$S_j = U; \Box X; \neg\Box\overline{Y_j}; \Box\neg\Box Q_j$

$S_{k+j} = \Box X; \neg Q_j; \Box(Q_j \Rightarrow \Box Q_j); \neg\Box\overline{Y_j}$

for $1 \leq j \leq k$

Figure 4: Tableau rules $(S4.14)$, $(S4.3)$ and $(S4.3.1)$.

The logical rules are categorised into two sorts, static rules or transitional rules, as follows:

Static Rules	Transitional Rules
(0), $(\neg)$, $(\wedge)$, $(\vee)$, (T)	$(S4)$, $(S4.3)$, $(S4.3.1)$, $(S4.14)$

The tableau method is a search for a counter model. The intuition behind this sorting is that in the static rules, the numerator and denominator represent the same world, whereas in the transitional rules, the numerator and denominator represent different worlds.

The tableau calculi $\mathcal{C}S4$, $\mathcal{C}S4.3$, $\mathcal{C}S4.3.1$ and $\mathcal{C}S4.14$ are respectively the calculi for the logics **S4**, **S4.3**, **S4.3.1** and **S4.14** as shown below:

$\mathcal{CL}$	Static Rules	Transitional Rules	Structural Rules
$\mathcal{C}S4$	$(0), (\neg), (\wedge), (\vee), (T)$	$(S4)$	(θ)
$\mathcal{C}S4.3$	$(0), (\neg), (\wedge), (\vee), (T)$	$(S4.3)$	(θ)
$\mathcal{C}S4.3.1$	$(0), (\neg), (\wedge), (\vee), (T)$	$(S4), (S4.3.1)$	(θ)
$\mathcal{C}S4.14$	$(0), (\neg), (\wedge), (\vee), (T)$	$(S4), (S4.14)$	(θ)

Note that $\mathcal{C}S4.3$ does not contain the rule $(S4)$ and that $\mathcal{C}S4.3.1$ does not contain the rule $(S4.3)$ but does contain the rule $(S4)$.

The subformula property for tableau systems in primitive notation is slightly different than that for sequent systems where the left side and right side of the sequent arrow respectively act as signs representing "true" and "false". In fact, Fitting makes these signs explicit in his signed tableau [4]. In our tableau systems, the formulae from the right side of the sequent $\Gamma \longrightarrow \Delta$ appear with an extra negation sign in the tableau node carrying $\Gamma \cup \neg\Delta$. Hence the "subformulae" we need to consider in our tableaux must contain the negated versions of the sequent subformulae. The following definitions cater for this change. For any finite set X :

- let $Sf(X)$ denote the set of all subformulae of all formulae in X ;
- let $\neg Sf(X)$ denote $\{\neg P \mid P \in Sf(X)\}$;
- let $\widetilde{X}$ denote the set $Sf(X) \cup \neg Sf(X) \cup \{0\}$;
- let $\Box(\widetilde{X} \Rightarrow \Box\widetilde{X})$ denote the set $\{\Box(P \Rightarrow \Box P) \mid P \in \widetilde{X}\}$;
- let $X^*_{S4} = X^*_{S4.3} = \widetilde{X}$;
- let $X^*_{S4.3.1} = X^*_{S4.14} = Sf(\Box(\widetilde{X} \Rightarrow \Box\widetilde{X}))$.

Thus, a tableau system $\mathcal{C}L$ has the subformula property if every $\mathcal{C}L$-tableau rule maps a subset of X_L^* onto another subset of X_L^* and if $X_L^* = \widetilde{X}$. The tableau systems $\mathcal{C}S4.3.1$ and $\mathcal{C}S4.14$ do not have the subformula property, but given some finite X, the set X_L^* is always bounded, so that the "superformulae" that may appear in a tableau node are bounded. We call this an **analytical superformula** property and formalise this with the following lemma.

Lemma 2 *If there is a closed $\mathcal{C}L$ tableau for the finite set X then there is a closed $\mathcal{C}L$ tableau for X with all nodes in the finite set X_L^*.*

Proof: Obvious from the fact that all rules for $\mathcal{C}L$ operate with subsets of X_L^* only. •

A set X is **closed with respect to a tableau rule** if, whenever (an instantiation of) the numerator of the rule is in X, so is (a corresponding instantiation of) at least one of the denominators of the rule. A set X is **$\mathcal{C}L$-saturated** if it is $\mathcal{C}L$-consistent and closed with respect to the static rules of $\mathcal{C}L$. That is, with respect to each of $(0), (\neg), (\wedge), (\vee)$, and (T).

Lemma 3 *For each $\mathcal{C}L$-consistent X there is an effective procedure to construct some finite $\mathcal{C}L$-saturated X^* with $X \subseteq X^* \subseteq X_L^*$.*

Proof: Since X is $\mathcal{C}L$-consistent, we know that no $\mathcal{C}L$-tableau for X closes and hence that the (0) rule is not applicable. So any (static) rule we apply is guaranteed to give at least one $\mathcal{C}L$-consistent set and we can form a sequence of $\mathcal{C}L$-consistent sets $X_0 = X, X_1, X_2, \cdots$ If applying some rule to X_i gives a previous member of the sequence then we backtrack and choose a different rule to apply to X_i. For $(\vee)$ we first decide which denominator is $\mathcal{C}L$-consistent and choose the corresponding denominator as the next set in the sequence. This procedure will terminate with some final X_n either because all rule applications lead to a cycle or because no rule is applicable to X_n. Put $X^* = X_0 \cup X_1 \cup X_2 \cup \cdots \cup X_n$ giving a $\mathcal{C}L$-saturated set X^*. Since each rule carries subsets of X_L^* to subsets of X_L^* and we start with $X \subseteq X_L^*$ we have $X \subseteq X^* \subseteq X_L^*$. •

The following definition from Rautenberg [10] is central for the model constructions. A **model graph** for some finite fixed set of formulae X is a finite **L**-frame $\langle W, R\rangle$ such that all $w \in W$ are $\mathcal{C}L$-saturated sets with $w \subseteq X_L^*$ and

(i) $X \subseteq w_0$ for some $w_0 \in W$;

(ii) if $\neg\Box P \in w$ then there exists some $w' \in W$ with wRw' and $\neg P \in w'$;

(iii) if wRw' and $\Box P \in w$ then $P \in w'$.

Lemma 4 *If $\langle W, R \rangle$ is a model graph for X then there exists an* **L***-model for X [10].*

Proof: For every $p \in \mathcal{P}$, let $\vartheta(p) = \{w \in W \mid p \in w\}$. Using simultaneous induction on the degree of $A \in w$ it is easy to show that (a) $A \in w$ implies $w \models A$ and (b) $\neg A \in w$ implies $w \not\models A$. By (a), $w_0 \models X$ hence $\langle W, R, \vartheta \rangle$ is an **L**-model for X [10]. •

This model graph construction is similar to the subordinate frames construction of Hughes and Cresswell [8] except that Hughes and Cresswell use maximal consistent sets and do not consider cycles, giving infinite models rather than finite models.

3 Soundness of $\mathcal{CL}$

A formula $\neg\Box P$ is called an **eventuality** since it entails that eventually $\neg P$ must hold. A world w is said to **fulfill** an eventuality $\neg\Box P$ when $w \models \neg P$. A sequence of worlds $w_1 R w_2 R \cdots R w_m$ is said to fulfill an eventuality $\neg\Box P$ when $w_i \models \neg P$ for some w_i in the sequence.

Theorem 1 *The $\mathcal{CL}$ rules are sound with respect to* **L***-frames.*

For each rule in $\mathcal{CL}$ we have to show that if the numerator of the rule is **L**-satisfiable then so is at least one of the denominators. The only interesting cases are the proofs for the modal rules.

Proof for (T)**:** Follows from the fact that all **L**-models are reflexive.

Proof for $(S4)$**:** This follows from the semantics for $\neg\Box P$ as "eventually there is a world where P is false"; from the guaranteed seriality of R for **S4**-models by the reflexivity of R; and from the transitivity of R for **S4**-models. The $(S4)$ rule can be seen as a "jump" to the world where $\neg P$ eventually becomes true [4]. The same argument shows that $(S4)$ is sound for **S4.3.1**-frames and for **S4.14**-frames.

Proof for $(S4.3)$**:** The $(S4.3)$ rule is based on a consequence of the characteristic **S4.3** axiom 3. Adding 3 to **S4** gives a weakly-connected R for **S4.3** so that eventualities can be weakly-ordered. If there are k eventualities, one of them must be fulfilled first. The $(S4.3)$ rule can be seen as a disjunctive choice between which one of the k eventualities is fulfilled first and an appropriate "jump" to the corresponding world.

An axiomatic argument is that the following is a theorem of **S4.3** [18, page 232-233]:

$$\neg\Box P \wedge \neg\Box Q \Rightarrow \Diamond(\neg\Box P \wedge \neg Q) \vee \Diamond(\neg\Box Q \wedge \neg P).$$

The soundness of the $(S4.3)$ rule follows from a generalised version of this last **S4.3**-theorem containing k formulae of the form $\neg\Box P_1 \cdots \neg\Box P_k$ [18, pages 236-238].

Proof for ($S4.3.1$)**:** By the law of the excluded middle, $\Box\neg\Box P \vee \neg\Box\neg\Box P$ is **L**-valid. So each eventuality is either an invariant $\Box\neg\Box P$ or there is a point in the future where $\Box P$ becomes true. In the latter case, the truth value of P eventually settles down to "true". The notion of "unique predecessor" is well-defined in **S4.3.1**-frames and the unique world immediately preceding this point satisfies $\Box(P \Rightarrow \Box P) \wedge \neg P$. But if there is more than one eventuality in the numerator, then any of these may settle down first and the ($S4.3.1$) rule must cater to these orderings in the same way as did the ($S4.3$) rule.

That is, the S_i denominators "assume" that $\neg\Box P$ is an invariant by lifting it to $\Box\neg\Box P$. The S_{k+i} denominators make the opposite assumption that $\Box\neg\Box P$ is false; that is, that $\Diamond\Box P$ is true. But we cannot simply "lift" $\neg\Box P$ to $\neg\Box\neg\Box P$ for then the ($S4.3.1$) rule would no longer be analytic as the eventuality $\neg\Box P$ would spawn the eventuality $\neg\Box\neg\Box P$ which could then spawn another eventuality $\neg\Box\neg\Box\neg\Box P$ and so on.

Proof for ($S4.14$)**:** By the law of the excluded middle, $\Diamond\Box\neg\Box P \vee \neg\Diamond\Box\neg\Box P$ is **L**-valid. That is, either there is some branch on which the value of P never settles to "true", or the value of P settles to "true" down every branch. In the latter case, the unique parent of this node satisfies $\Box(P \Rightarrow \Box P) \wedge \neg P$. $\bullet$

The ($S4.3.1$) and ($S4.14$) rules can also be motivated by rewriting Dum as

$$\neg\Box P \Rightarrow \Diamond(\Box(P \Rightarrow \Box P) \wedge \neg P) \vee \Box\neg\Box P$$

and rewriting Zbr as

$$\neg\Box A \Rightarrow \Diamond\Box\neg\Box A \vee \Diamond(\Box(A \Rightarrow \Box A) \wedge \neg A)$$

For example, the left fork of the ($S4.14$) rule is a jump to the world where $\Box\neg\Box A$ eventually becomes true and the right fork is a jump to the world where $\Box(A \Rightarrow \Box A) \wedge \neg A$ eventually becomes true.

4 Completeness of $\mathcal{C}L$

In this section we shall construct model-graphs using $\mathcal{C}L$-saturated sets and an immediate successor relation $\prec$. We therefore extend the notion of fulfilling an eventuality. A set w is said to fulfill an eventuality $\neg\Box P$ when $\neg P \in w$. A sequence $w_1 \prec w_2 \prec \cdots \prec w_m$ of sets is said to fulfill an eventuality $\neg\Box P$ when $\neg P \in w_i$ for some w_i in the sequence.

Theorem 2 *If X is a finite set of formulae and X is $\mathcal{C}L$-consistent then there is an* **L***-model for X on a finite* **L***-frame $\langle W, R\rangle$.*

Proof for $\mathcal{C}S4$**:** The construction of the model graph is due to Rautenberg [10] where $\prec$ denotes the immediate successor relation. By Lemma 3 (page 8) we can construct some $\mathcal{C}S4$-saturated $X^* = w_0$ with $X \subseteq w_0 \subseteq X^*_{S4}$. If no $\neg\Box P$ occurs in w_0 then $\langle\{w_0\}, \{(w_0, w_0)\}\rangle$ is the desired model graph since it is an **S4**-frame and (i)-(iii) are satisfied. Otherwise, let $Q_1, Q_2, \cdots Q_m$ be all the formulae such that $\neg\Box Q_i \in w_0$ and $\neg Q_i \notin w_0$.

Let $w' = \{P | \Box P \in w_0\}$. Since $\Box w' \subseteq w_0$, we know that $\Box w' \cup \{\neg\Box Q_i\}$ is $\mathcal{C}S4$-consistent by (θ); hence so is each $X_i = \Box w' \cup \{\neg Q_i\}, \ \ i = 1, \cdots, m$ by $(S4)$.

For each X_i we can find some $\mathcal{C}S4$-saturated v_i with $X_i \subseteq v_i \subseteq X^*_{S4}$ by Lemma 3. Put $w_0 \prec v_i$, $i = 1, \cdots, m$ and call v_i the Q_i-successor of w_0. These are the immediate successors of w_0. Now repeat the construction with each v_i thus obtaining the nodes of level 2 and so on.

In general, the above construction of $\langle W, \prec\rangle$ runs ad infinitum. However, since $w \in W$ implies $w \subseteq X^*_{S4}$, a sequence $w_0 \prec w_1 \prec \cdots$ in $\langle W, \prec\rangle$ either terminates, or a node repeats. If in the latter case $n > m$ are minimal with $w_n = w_m$ we stop the construction and identify w_n and w_m in $\langle W, \prec\rangle$ thus obtaining a circle instead of an infinite path. One readily confirms that $\langle W, R\rangle$ is a model graph for X where R is the reflexive and transitive closure of $\prec$. It is obvious that clusters in $\langle W, R\rangle$ form a tree.

Now $\langle W, R\rangle$ is an **S4**-model graph for X so by Lemma 4 (page 9), there exists an **S4**-model $\langle W, R, \vartheta\rangle$ which is an **S4**-model for X where $\vartheta : p \mapsto \{w \in W \mid p \in w\}$.

Note that, when creating successors for some w_n, the proof for $\mathcal{C}S4$ still goes through if we let $Q_1, Q_2, \cdots Q_m$ be *all* the formulae such that $\neg\Box Q_i \in w_n$, thus creating unnecessary successors for the eventualities fulfilled by w_n itself. The proof given above creates a smaller model since we pre-empt the reflexivity of R. •

Proof sketch for $\mathcal{C}S4.3$: The completeness proof of $\mathcal{C}S4.3$ is similar to the completeness proof for $\mathcal{C}S4$. The differences are that only *one* sequence is constructed, and that in doing so, the $(S4.3)$ rule is used instead of the $(S4)$ rule. Note that the $(S4.3)$ rule guarantees only that *at least one* eventuality gives a $\mathcal{C}S4.3$-consistent successor whereas $(S4)$ guarantees that *every* eventuality gives a $\mathcal{C}S4$-consistent successor. The basic idea is to follow one sequence, always attempting to choose a successor new to the sequence. Sooner or later, no such successor will be possible giving a sequence $S = w_0 \prec w_1 \prec w_2 \prec \cdots \prec w_m \prec w_{m+1} \prec \cdots \prec w_{n-1} \prec w_m$ containing a cycle $C = w_m \prec w_{m+1} \prec \cdots \prec w_{n-1} \prec w_m$ which we write pictorially as

$$S = w_0 \prec w_1 \prec w_2 \prec \cdots \prec \overline{w_m \prec w_{m+1} \cdots \prec w_{n-1}}.$$

The cycle C fulfills at least one of the eventualities in w_{n-1}, namely the $\neg\Box Q$ that gave the duplicated Q-successor w_m of w_{n-1}. But C may not fulfill *all* the eventualities in w_{n-1}.

Let $Y = \{P \mid \neg\Box P \in w_{n-1}$ and $\neg P \notin w_j, \ m \leq j \leq n-1\}$, so that $\neg\Box Y$ is the set of eventualities in w_{n-1} that remain unfulfilled by C. Let $w' = \{P \mid \Box P \in w_{n-1}\}$. Since

$(\Box w'; \neg\Box Y) \subseteq w_{n-1}$ is $\mathcal{C}S4.3$-consistent by (θ), so is at least *one* of

$$X_j = \Box w' \cup \{\neg P_j\} \cup \neg\Box\overline{Y_j}, \text{ for } j = 1, \cdots, k$$

by $(S4.3)$. As before, choose the $\mathcal{C}S4.3$-consistent X_i that gives a **S4.3**-saturated P_i-successor for w_{n-1} which is new to S to sprout a continuation of the sequence, thus escaping out of the cycle. If no such new successor is possible then choose the successor $w_{m'}$ that appears earliest in S. This successor *must* precede w_m, as otherwise, C would already fulfill the eventuality that gives this successor. That is, we can extend C by putting $w_{n-1} \prec w_{m'}$. Recomputing Y using m' instead of m must decrease the size of Y since w_{n-1} has remained fixed. Repeating this procedure will eventually lead either to an empty Y or to a new successor. In the latter case we carry on the construction of S. In the former case we form a final cycle that fulfills all the eventualities of w_{n-1} and stop.

Sooner or later we must run out of new successors since $X^*_{S4.3}$ is finite and so only the former case is available to us. Let R be the reflexive and transitive closure of $\prec$ so that the overlapping clusters of $\prec$ become maximal disjoint clusters of R. It should be clear that $\langle W, R\rangle$ is a linear order of maximal, disjoint clusters that satisfies properties (i)-(iii), and hence that $\langle W, R\rangle$ is a model-graph for X.

Note that thinning seems essential. That is we *have* to exclude the eventualities that are already fulfilled by the current cycle C in order to escape out of the cycle that they cause. We return to this point later. •

Proof Sketch for $\mathcal{C}S4.3.1$: If w_0 contains no eventualities then $\langle\{w_0\}, \{(w_0, w_0)\}\rangle$ is the desired model graph. Otherwise, let the current sequence be $S = w_0$ with $n = 0$ in the steps below. In general, w_n always denotes the last node in $S = w_0 \prec w_1 \prec \cdots \prec w_{n-1} \prec w_n$.

Step 0: Let $Y_n = \{\neg\Box P \in w_n \mid \neg P \notin w_n \text{ and } \Box\neg\Box P \notin w_n\}$. These are the unfulfilled and non-invariant eventualities of w_n. If Y_n is empty then go to Step 2, otherwise do Step 1.

Step 1: Let $w' = \{Q \mid \Box Q \in w_n\}$. Assuming that Y_n is not empty, we know that $\Box w' \cup Y_n$ is $\mathcal{C}S4.3.1$-consistent by (θ). Hence there is a successor due to $(S4.3.1)$ for some $\neg\Box P \in Y_n$.

If the successor is due to one of the denominators $S_1, \cdots, S_k$ then call it w'_n and replace w_n by w'_n giving $S = w_0 \prec w_1 \prec \cdots \prec w_{n-1} \prec w'_n$ and return to Step 0 with n unchanged.

Else, the successor is due to one of the denominators $S_{k+1}, \cdots, S_{k+k}$ so call it w_{n+1} and put $w_n \prec w_{n+1}$ giving $S = w_0 \prec w_1 \prec \cdots \prec w_{n-1} \prec w_n \prec w_{n+1}$. Increment n and go to Step 0; that is, w_{n+1} is now the last node of S.

Now, $\neg\Box P \in Y_n$ and w'_n contains $\Box\neg\Box P$. Furthermore, no member of S contains $\Box\neg\Box P$ since that implies $\Box\neg\Box P \in w_n$ which in turn implies $\neg\Box P \notin Y_n$. Hence w'_n is new to S. But note that $w_n \subseteq w'_n$ since the denominators $S_1, \cdots S_k$ of the $(S4.3.1)$ rule merely lift some $\neg\Box P$ to $\Box\neg\Box P$ and we regain $\neg\Box P$ by (T), so we lose no formulae of w_n.

Alternatively, the successor w_{n+1} contains $\Box(P \Rightarrow \Box P)$ and $\neg P$ for some eventuality $\neg\Box P$ of Y_n. Suppose w_{n+1} duplicated some existing node of S. Then w_n would contain $(P \Rightarrow \Box P)$ by (T) because both $(S4)$ and $(S4.3.1)$ preserve $\Box$-formulae. But $(P \Rightarrow \Box P)$ is just abbreviation for $\neg(P \wedge \neg\Box P)$, hence by $(\vee)$, w_n contains $\neg P$ or $\neg\neg\Box P$. Since $\neg\Box P \in Y_n$, the first is impossible. And the second contradicts the $\mathcal{C}S4.3.1$-consistency of w_n since $\neg\Box P \in w_n$. Thus, w_{n+1} must be new to S.

Step 2: We know that $Y_n = \{\neg\Box P \in w_n \mid \neg P \notin w_n \text{ and } \Box\neg\Box P \notin w_n\}$ is empty. That is, for each $\neg\Box P \in w_n$, we have $\neg P \in w_n$ or $\Box\neg\Box P \in w_n$.

If $\neg\Box P \in w_n$ implies $\neg P \in w_n$ for all eventualities in w_n then we are done. Otherwise, let $Z = \{\neg\Box P \in w_n \mid \neg P \notin w_n \text{ and } \Box\neg\Box P \in w_n\}$. These are the unfulfilled eventualities of w_n. By (θ) and $(S4)$ *each* eventuality in Z has a $\mathcal{C}S4.3.1$-consistent successor. Choose any successor w_{n+1} that is new to S and put $w_n \prec w_{n+1}$. Since $\Box Z \subseteq w_n$ by definition of Z, the unfulfilled eventualities of w_n are carried into w_{n+1}; that is, $Z \subseteq w_{n+1}$. Increment n, and go to Step 0.

If no new successor is possible then choose the successor w_x appearing earliest in S and put $w_n \prec w_x$ to give a final cycle that fulfills all the eventualities of w_n and stop.

Sooner or later, we must run out of new successors since $X^*_{S4.3.1}$ is finite, or encounter a node w_n that fulfills all its own eventualities. Let R be the reflexive and transitive closure of $\prec$ to give an **S4.3.1**-frame $\langle W, R\rangle$.

Again, (θ) seems essential because we have to ignore some of the eventualities of w_n in selecting Y_n; namely the eventualities $\neg\Box P$ with $\Box\neg\Box P \in w_n$ or $\neg P \in w_n$. •

Proof for $\mathcal{C}S4.14$: Let $w_0 \supseteq X$ be $\mathcal{C}S4.14$-saturated, $w_0 \subseteq X^*_{S4.14}$. Construct a model graph from w_0 using the method for $\mathcal{C}S4$ except for one additional step. In general, when a Q_i-successor is created for $\neg\Box Q_i \in w$ (with $\neg Q_i \notin w$) based on the $(S4)$ rule and where $w' = \{P \mid \Box P \in w\}$, the additional rule $(S4.14)$ means that

(a) $\Box w' \cup \{\Box\neg\Box Q_i\}$ is $\mathcal{C}S4.14$-consistent or

(b) $\Box w' \cup \{\Box(Q_i \Rightarrow \Box Q_i), \neg Q_i\}$ is $\mathcal{C}S4.14$-consistent.

So each node can have a Q_i-successor due to $(S4)$ and at least one Q_i-successor due to $(S4.14)$. Note that the $(S4.14)$ rule denominators are not mutually exclusive so they can both be **S4.14**-consistent at the same time.

The construction still gives a preorder over $\prec$ as for $\mathcal{C}S4$ and each branch either terminates, or gives a cycle due to the finiteness of $X^*_{S4.14}$ by choosing the minimum i and j such that $w_i = w_j$, $j > i$ and putting $w_{j-1} \prec w_i$.

As for $\mathcal{C}S4$ let R be the reflexive and transitive closure of $\prec$ giving a finite model graph $\mathcal{F} = \langle W, R \rangle$ whose clusters form a tree. The graph may not be an **S4.14**-frame because **S4.14**-frames must not contain non-final proper clusters and this is not guaranteed of the graph $\langle W, R \rangle$. We claim that all non-final proper clusters can be eliminated from $\mathcal{F}$ whilst still preserving properties (i)-(iii) giving a model graph $\mathcal{F}'$.

To see this first note that $\mathcal{F}$ is a finite tree where each branch is a sequence of nondegenerate clusters of R. Suppose $C = w_1 R w_2 R w_3 R \cdots R w_n R w_1$, $n \geq 2$, is some arbitrary non-final proper cluster on some arbitrary branch in $\mathcal{F}$. Since each w_i is a member of a non-final cluster, each w_i must contain at least one (originally unfulfilled) eventuality. Consider some such $\neg\Box P \in w_n$. By $(S4.14)$, w_n has a successor w' that contains $\{\Box\neg\Box P\}$ or $\{\Box(P \Rightarrow \Box P), \neg P\}$.

If w' contains $\{\Box\neg\Box P\}$ then, regardless of whether w' occurs *in* C or *after* C, all worlds in some final cluster C_f reachable from C must contain $\Box\neg\Box P$ and hence must contain $\neg\Box P$ by (T). Since C_f is a *final* cluster, there must be some world in C_f that fulfills $\neg\Box P$.

If w' contains $\{\Box(P \Rightarrow \Box P), \neg P\}$ then w' must occur *strictly after* w_n. For otherwise, by the transitivity of R we would have $\Box(P \Rightarrow \Box P) \in w_n$ and hence $(P \Rightarrow \Box P) \in w_n$ by (T). But $(P \Rightarrow \Box P)$ is just abbreviation for $\neg P \vee \Box P$ hence we would have $\neg P \in w_n$ or $\Box P \in w_n$ by $(\vee)$. The former is impossible since we create a successor for $\neg\Box P$ only if $\neg P \notin w_n$. And the latter contradicts our supposition that $\neg\Box P \in w_n$.

Thus we can liberate w_n from C without breaking properties (ii) and (iii) giving two consecutive nondegenerate clusters $C' = w_1 R w_2 R \cdots R w_{n-1} R w_1$ and $C_n = w_n R w_n$ so that C_n is simple. Applying the same arguments to C' allows us to liberate w_{n-1}, and so on, giving a linear sequence of simple clusters $C_1 R C_2 R \cdots R C_n$ where $C_i = w_i R w_i$.

As C was any non-final proper cluster, this can be done for all non-final proper clusters giving some final $\mathcal{F}' = \langle W, R' \rangle$ that is also a model graph where R' is the altered reachability relation. But $\mathcal{F}'$ is now an **S4.14**-frame since it contains no non-final proper clusters.

Property (i) still holds because we have not removed any elements of W hence $X \subseteq w_0 \subseteq W$. Property (iii) holds because we have not added any extra tuples to R, only removed some. So if it held before the pruning process, it must hold after it. And property (ii) holds because of the argument above. Since properties (i)-(iii) still hold, $\mathcal{F}'$ is also a model graph for X.

Note that the proof does not stipulate any particular ordering for $C = w_1 R \cdots R w_n R w_1$. That is, C can be flattened into an arbitrary sequence of its constituent worlds and consequently, the proof is constructive. •

5 Decision Procedures for L

In each tableau system $\mathcal{C}L$, each tableau node is labelled with a finite subset of X_L^*. Any node (subset of X_L^*) that reappears on a branch is an end node of the tableau by definition. Thus, since each X_L^* is finite, there are a finite number of different $\mathcal{C}L$-tableaux for any given finite X. If any one of them closes then X has no **L**-model by the soundness of $\mathcal{C}L$. If no $\mathcal{C}L$-tableau closes then we can construct a finite **L**-model for X via the completeness proof. Therefore, each $\mathcal{C}L$ is a highly nondeterministic decision procedure for each **L**.

There is, however, a completely different deterministic decision procedure for **L** in the $\mathcal{C}L$ completeness proof since each completeness proof is constructive, and hence is an effective **L**-satisfiability test. That is, it is a (deterministic) procedure which uses $\mathcal{C}L$-saturated sets to construct a finite **L**-model for some finite set X. The deterministic decision procedure described above is the basis of most decision procedures for temporal logics as exemplified by those of Wolper [17].

6 Admissibility of Cut and Sequent Systems

The cut rule is sound with respect to all our **L**-frames and each $\mathcal{C}L$ is sound and complete with respect to the appropriate **L**-frames. Thus, putting (ρ) equal to (cut) in Lemma 1 gives:

Theorem 3 *The rule* (cut) *is admissible in each* $\mathcal{C}L$.

Figure 5 shows the sequent analogues of each of our tableau rules. Each (cut-free) tableau system therefore has a (cut-free) sequent analogue defining a finitary provability relation $\vdash_L$ for each axiomatically formulated logic **L**. Consequently, any tableau proof can be converted into a sequent proof which can be converted into an axiomatic proof. In particular, the cut rule, which in our sequent formulation has the form

$$(cut)\ \frac{\Gamma \longrightarrow A, \Delta \quad \Gamma, A \longrightarrow \Delta}{\Gamma \longrightarrow \Delta}$$

is redundant by the above theorem on admissibility.

From now on, we speak of $\mathcal{C}L$ as a tableau or sequent system depending on the point we wish to stress.

Our sequent systems do not possess all the elegant properties usually demanded of (Gentzen) sequent systems. For example, not only do some of our systems break the subformula property, but most do not possess clear rules for introducing modalities into the right and left sides of sequents.

$$\frac{\Box\Gamma \longrightarrow \ \Diamond\Box A \qquad\qquad \Box\Gamma,\ \Box(A \Rightarrow \Box A) \longrightarrow A}{\Box\Gamma \longrightarrow \Box A}\ (\rightarrow \Box : S4.14)$$

$$\frac{S_1 \qquad S_2 \qquad \cdots \qquad S_k}{\Box\Gamma \longrightarrow \Box A_1, \cdots, \Box A_k}\ (\rightarrow \Box S4.3)$$

where for $1 \leq i \leq k$

$$Y = \{A_1, \cdots, A_k\}$$

$$\overline{Y_i} = Y \setminus \{A_i\}$$

$$S_i = \Box\Gamma \longrightarrow A_i,\ \Box\overline{Y_i}$$

$$\frac{S_1 \quad S_2 \quad \cdots \quad S_k \quad S_{k+1} \quad S_{k+2} \quad \cdots \quad S_{2k}}{\Sigma,\ \Box\Gamma \longrightarrow \Box A_1, \cdots, \Box A_k,\ \Delta}\ (\rightarrow \Box : S4.3.1)$$

where for $1 \leq i \leq k$

$$Y = \{A_1, \cdots, A_k\}$$

$$\overline{Y_i} = Y \setminus \{A_i\}$$

$$S_i = \Sigma,\ \Box\Gamma \ \longrightarrow\ \Diamond\Box A_i,\ \Box\overline{Y_i},\ \Delta$$

$$S_{k+i} = \Box\Gamma,\ \Box(A_i \Rightarrow \Box A_i) \ \longrightarrow\ A_i,\ \Box\overline{Y_i}$$

Figure 5: Sequent analogues of tableau rules $(S4.14)$, $(S4.3)$ and $(S4.3.1)$.

7 Eliminating Thinning

The structural rule (θ) corresponds to the sequent rule of weakening which explicitly enforces monotonicity. From a theorem proving perspective, (θ) introduces a form of nondeterminism into each $\mathcal{C}L$ since we have to guess which formula are really necessary for a proof. It is therefore desirable to eliminate (θ). There are two places where we resort to applications of (θ) in our completeness proofs. We consider each in turn.

In all the completeness proofs we avoid creating a successor for $\neg\Box Q \in w$ if $\neg Q \in w$, thus using (θ) to pre-empt reflexivity of R. This is not an *essential* application of (θ) in $\mathcal{C}S4$ and $\mathcal{C}S4.14$ because a consistent successor also exists for these eventualities, it is just that we are not interested in these successors.

The only other applications of (θ) in $\mathcal{C}S4$ and $\mathcal{C}S4.14$ are the ones used to eliminate all non-boxed formulae prior to an application of a transitional rule. The crucial point is that we know exactly which formulae to throw away: namely, the non-boxed ones. Consequently, (θ) can be eliminated by building thinning in a deterministic way into the transitional rules $(S4)$ and $(S4.14)$. For example, we can change

$$(S4)\ \frac{\Box X; \neg\Box P}{\Box X; \neg P} \qquad \text{to} \qquad (S4')\ \frac{X; \neg\Box P}{X'; \neg P}$$

with $X' = \{\Box Q \mid \Box Q \in X\}$ and simultaneously changing the basic axiomatic tableau rule from

$$(0)\ \frac{P; \neg P}{0} \qquad \text{to} \qquad (0')\ \frac{X; P; \neg P}{0}$$

see Fitting [4].

In terms of sequent systems this just says that a given proof of $\Gamma \longrightarrow \Delta$ in $\mathcal{C}L$ can be converted into a proof of $\Gamma \longrightarrow \Delta$ in $\mathcal{C}L'$ either by moving applications of weakening into the (new) axiomatic leaf sequents, or into the (new) transitional rules (in a deterministic way).

However, (θ) appears essential for **S4.3** and **S4.3.1**. First, note that the $(S4.3)$ rule can simulate the $(S4)$ rule by thinning Y in Figure 4 (page 6) to be a singleton. Second, note that the $(S4.3)$ rule is not a rule of $\mathcal{C}S4.3.1$; however, it is not derivable in $\mathcal{C}S4.3.1$ either. The $(S4.3)$ rule is odd in that more than one eventuality plays an active role in any one rule application. That is, if $\{\neg\Box P_1, \cdots, \neg\Box P_m\}$ are all the eventualities in the current tableau proof node, then by appropriate uses of (θ) we may choose Y in Figure 4 (page 6) to be *any* non-empty subset of $\{P_1, \cdots, P_m\}$.

In the counter-model construction for $\mathcal{C}S4.3$, we may reach a stage where all $\mathcal{C}S4.3$-consistent successors already appear in S but no such cycle fulfills all the eventualities of the last node. At this stage it is essential to invoke applications of (θ) on subsets of the eventualities. That is, we must be able to *ignore* some of the eventualities in w_{n-1} using (θ) and this means that (θ) is now an essential rule of $\mathcal{C}S4.3$.

Similarly, in the completeness proof of $\mathcal{C}S4.3.1$, it is essential to ignore $\neg\Box P$ if $\neg P \in w$ or if $\Box\neg\Box P \in w$ in order to guarantee that the ensuing successor is new to the sequence, and again thinning seems essential.

It may be possible to eliminate thinning by using cleverer completeness proofs. For example, an alternate proof for $\mathcal{C}S4.3$ may be possible by considering all $(S4.3)$-successors for every node, giving a tree of nondegenerate clusters, and then showing that any two worlds in this tree can be ordered as is done by Hughes and Cresswell [8, page 30-31]. Note however that this seems to require a cut rule since Hughes and Cresswell use maximal consistent sets rather than saturated sets as we do.

Clearly the intuitions inherent in our semantic methods are no longer enough to prove that weakening is eliminable. We have obtained a syntactic proof of elimination of weakening in the sequent system corresponding to $\mathcal{C}S4.3'$, the system containing the modified rules $(S4.3')$ and $(0')$, but this is beyond the scope of this article.

8 Eliminating Contraction

When formulated using sets, the rule of contraction

$$\frac{X; P}{X; P; P}$$

is hidden in the notation since the sets $X; P$ and $X; P; P$ are the same. Some of the tableau rules we have given are not standard; for example, the (T) rule is usually given as:

$$\frac{X; \Box P}{X; P}$$

where $\Box P$ is not carried from the numerator into the denominator [10]. It is well known that the rule of contraction, which is implicit in the set formulation, then becomes *essential* for completeness. It is also well known that although contraction becomes essential, it is required *only* for $\Box$-formulae in most normal modal logics, and on both $\Box$-formulae and $\Diamond$-formulae in some symmetric normal modal logics [5]. We have deliberately built contraction into our rules to highlight this fact. We believe that if we interpret ";" as multiset union, and rework our formulation using multisets instead of sets, then all the proofs will still go through with appropriate modifications. That is, the rule of contraction appears to be *eliminable* from our systems as long as the static rules build in contraction as given by our rules. Unfortunately, the proofs become very messy.

9 Further Work

The logics **S4.3** and **S4.3.1** respectively have counterparts called **K4DLX** and **K4DLZ** [6] that omit reflexivity where the new axiom schema are:

$$
\begin{array}{ll}
D & \Box A \Rightarrow \Diamond A; \\
L & \Box((A \wedge \Box A) \Rightarrow B) \vee \Box((B \wedge \Box B) \Rightarrow A); \\
X & \Box\Box A \Rightarrow \Box A; \text{ and} \\
Z & \Box(\Box A \Rightarrow A) \Rightarrow (\Diamond\Box A \Rightarrow \Box A).
\end{array}
$$

It is known that

$$\langle \mathcal{I}, < \rangle \models A \text{ iff } \vdash_{K4DLX} A$$

and

$$\langle \omega, < \rangle \models A \text{ iff } \vdash_{K4DLZ} A$$

where $\mathcal{I}$ is either the set of real numbers or the set of rational numbers and ω is the set of natural numbers [6]. Hence these logics model irreflexive linear dense and irreflexive linear discrete time although the *finite* frames that characterise these logics are respectively finite sequences of finite (degenerate or nondegenerate) clusters with no consecutive degenerate clusters ; and finite sequences of finite (degenerate or nondegenerate) clusters with no proper non-final clusters. I am not aware of a proof of completeness for the non-reflexive counterpart of **S4.14** but it seems reasonable to conjecture that $\mathbf{K4DZ_{14}}$ is this counterpart where Z_{14} is:

$$Z_{14} \qquad \Box(\Box A \Rightarrow A) \Rightarrow (\Box\Diamond\Box A \Rightarrow \Box A).$$

The simplest way to handle the seriality axiom D is to use the static (D) rule of Rautenberg even though it breaks the subformula property. Another way is to allow the transitional rules to operate even when the numerator contains no eventualities. We follow the first approach for brevity; see Figure 6.

Another minor complication is the need for an explicit tableau rule to capture density (no consecutive degenerate clusters) for **K4DLX** but this is handled by the transitional rule $(K4LX)$, which is sound for **K4DLX**-frames.

The non-reflexive analogue of the $(S4.3)$ rule becomes very clumsy since it is based on the **K4LX**-theorem:

$$\Diamond P \wedge \Diamond Q \Rightarrow \Diamond(P \wedge \Diamond Q) \vee \Diamond(Q \wedge \Diamond P) \vee \Diamond(P \vee Q)$$

and it is easier to use the rule $(K4L)$ which makes explicit use of subsets. The $(K4L)$ rule is similar to a rule given by Valentini [15]. By using rules from Figure 6 it is possible to obtain cut-free tableau calculi possessing the analytic superformula property for these logics as:

$$(K4)\ \frac{\Box X; \neg\Box P}{X; \Box X; \neg P} \qquad (D)\ \frac{X; \Box P}{X; \Box P; \neg\Box\neg P}$$

$$(K4Z_{14})\ \frac{\Box X; \neg\Box P}{X; \Box X; \Box\neg\Box P \quad | \quad X; \Box X; \neg P; \Box P} \qquad (K4LX)\ \frac{\Box X; \neg\Box Y}{X; \Box X; \neg\Box Y}$$

$$(K4L)\ \frac{\Box X; \neg\Box\{P_1, \cdots, P_k\}}{X; \Box X; \neg\Box\overline{Y^i}; \neg Y^i}$$

for some i where

$Y^1, \cdots, Y^m$ is an enumeration of the non-empty subsets of Y

$m = 2^k - 1, \quad 1 \leq i \leq m$ and

$\overline{Y^i} = Y \setminus Y^i$

$$(K4LZ)\ \frac{U; \Box X; \neg\Box\{Q_1, \cdots, Q_k\}}{S_1 \ | \ S_2 \ | \ \cdots \ | \ S_k \ | \ S_{k+1} \ | \ S_{k+2} \ | \ \cdots \ | \ S_{k+m}}$$

where :

$Y = \{Q_1, \cdots, Q_k\}$;

$Y^1, \cdots, Y^m$ is an enumeration of the non-empty subsets of Y with $m = 2^k - 1$;

$\overline{Y_j} = Y \setminus \{Q_j\}$ for $1 \leq j \leq k$;

$\overline{Y^i} = Y \setminus Y^i$ for $1 \leq i \leq m$;

$S_j = U; \Box X; \neg\Box\overline{Y_j}; \Box\neg\Box Q_j$ for $1 \leq j \leq k$;

$S_{k+i} = X; \Box X; \neg Y^i; \Box Y^i; \neg\Box\overline{Y^i}$ for $1 \leq i \leq m$

Figure 6: Tableau rules $(K4)$, $(K4Z_{14})$, $(K4LX)$, $(K4L)$ and $(K4LZ)$.

$\mathcal{C}L$	Static Rules	Transitional Rules	Structural Rules
$\mathcal{C}K4D$	$(0), (\neg), (\wedge), (\vee), (D)$	$(K4)$	(θ)
$\mathcal{C}K4DLX$	$(0), (\neg), (\wedge), (\vee), (D)$	$(K4L), (K4LX)$	(θ)
$\mathcal{C}K4DLZ$	$(0), (\neg), (\wedge), (\vee), (D)$	$(K4), (K4LZ)$	(θ)
$\mathcal{C}K4DZ_{14}$	$(0), (\neg), (\wedge), (\vee), (D)$	$(K4), (K4Z_{14})$	(θ)

Now, it may appear as if the explicit subset notation would allow us to dispense with (θ) but this is not so. For (θ) allows us to *ignore* certain eventualities, whereas $(K4L)$ and $(K4LZ)$ only allow us to *delay* them. Thus using the reflexive analogues of these rules for **S4.3** and **S4.3.1** does not help to eliminate (θ).

Finally, these techniques extend easily to give a cut-free sequent system for **S4.3Grz** = **KGrz.3** [16] which is axiomatised as $K + 3 + Grz$ where Grz is the Grzegorczyk axiom schema Grz:$\Box(\Box(A \Rightarrow \Box A) \Rightarrow A) \Rightarrow A$. This logic is characterised by finite linear sequences of simple clusters but note that Shimura [14] has already given such a sequent system.

The non-reflexive counterpart of **S4.3Grz** is **KLG** (sometimes called **G.3** or $\mathbf{GL_{lin}}$ or **K4.3W**) where L is as above and G is the Gödel-Löb axiom $\Box(\Box A \Rightarrow A) \Rightarrow \Box A$. Rautenberg [10] shows that **KG** is characterised by the class of finite transitive trees of irreflexive worlds. Thus **KLG** is characterised by finite linear sequences of irreflexive worlds, but note that Valentini [15] has already given a cut-free sequent system for this logic.

10 Related Work

Zeman [18] appears to have been the first to give a tableau system for **S4.3** but he is unable to extract the corresponding cut-free sequent system [18, page 232]. Shimura [14] has given a syntactic proof of cut-elimination for the corresponding sequent system for **S4.3**, whereas we give a semantic proof. Apparently, Serebriannikov has also obtained this system for **S4.3** but I have been unable to trace this paper. Rautenberg [10] refers to "a simple tableau" system for **S4.3** but does not give details since his main interest is in proving interpolation, and **S4.3** lacks interpolation. In subsequent personal communications I have been unable to ascertain the **S4.3** system to which Rautenberg refers [12]. Bull [2] states that *"Zeman's* **Modal Logic** *(XLII 581), gives tableau systems for* **S4.3** *and* **D** *in its Chapter 15, ... "*. The **D** mentioned by Bull is **S4.3.1** but Zeman [18, page 245] merely shows that his tableau procedure for **S4.3** goes into unavoidable cycles when attempting to prove *Dum*. Zeman does not investigate remedies and consequently does *not* give a tableau system for **S4.3.1**. In fact, Bull [1] mentions that Kripke used semantic tableau for **S4.3.1**, in 1963, but he gives no reference and subsequent texts that use semantic tableau do not mention this work [18]. Presumably Kripke would have used tableaux where an explicit auxiliary relation is used to mimic the desired properties (like linearity) of R as is done in the semantic diagrams of Hughes and Cresswell [7, page 290]. Note that no such explicit representation of R is required in our systems where the desired properties of R are obtained by appropriate tableau rules. I know of no other (cut-free) sequent or tableau systems for the logics **S4.3.1** and **S4.14** or their non-reflexive counterparts **K4DLZ** and $\mathbf{K4DLZ_{14}}$.

11 Conclusions

We have presented cut-free tableau and sequent systems for the Diodorean modal logics **S4.3**, **S4.3.1** and **S4.14**, of which the last two appear to be new. We have also sketched how similar results for the non-reflexive counterparts of these logics can be obtained. The sequent analogues of our tableau systems give a finitary syntactic deducibility relation $\vdash_L$ so that any sequent proof can converted into an axiomatic proof of the endsequent. As a consequence, we obtain the admissibility of the cut rule for these systems. Each tableau system serves as a nondeterministic decision procedure for the logic it formulates. Furthermore, the proofs of tableau completeness are all constructive and yield deterministic decision procedures for each logic.

For some of our tableau systems, thinning seems essential. We believe that both thinning and contraction are eliminable by suitable modifications to the tableau rules but intend to pursue these matters using syntactic methods.

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