ISSN 1361 - 6161

THE UNIVERSITY
of MANCHESTER

Computer Science
University of Manchester

Notes on P-Algebra (2):
Group Presentation of Process Structure

Kohei Honda

Department of Computer Science

University of Manchester

Technical Report Series

UMCS-95-12-3

Notes on P-Algebra (2): Group Presentation of Process Structure*

Kohei Honda

Department of Computer Science
University of Manchester
Oxford Road, Manchester, UK.
khonda@cs.man.ac.uk

December 17, 1995

*Refereed by: Cliff Jones

Abstract

We give yet another presentation of the process structures introduced in [1] and prove its equivalence to the original one. The presentation enjoys several convenient features not found in the original presentation, though in compensation for a certain loss of simplicity. Among others, it dispenses with the use of a choice function for quotient construction. The new presentation offers a different viewpoint on the theory of process structure, and may provide a useful tool for some purposes.

1 Introduction

The present notes show that the process structures introduced in [1] can be given yet another presentation. Since a process structure is intended to provide a basic stratum of general process theory by giving a mathematical distillation of a basic element of concurrent processes (cf. [3]), its alternative presentation is interesting, because it offers a new way to look at the theory, and especially because different presentations are useful for different purposes.

The presentation we give in these notes is quite close to the "pure" presentation of process structure in [1], rather than to the "rooted" one also discussed there, and is perhaps more faithful to geometric truth: instead of looking at a process (think of it as a geometrical object) from a single direction, it tries to look at it from all possible directions. That is, the naming of handles, or ports, of each process is relativised by taking all their permutations. Thus the presentation of a process becomes a collection of permutation groups isomorphic to each other as group actions. One neat point is that it dispenses with the use of choice functions which is necessary in quotient and other constructions in the original presentation. At least set-theoretically this makes the whole theory cleaner. At the same time the group presentation loses some simplicity and concreteness of the original presentation, especially with respect to the designation of handles of a process. Thus, for the major development of theory from now on, the original one will be mainly used; however the new construction surely offers insight about the nature of the mathematical objects we are dealing with, and may provide us with a useful framework for an abstract treatment of the theory as well as for its possible ramifications.

In the rest of the notes, we describe basic elements of theory of process structure using a new presentation. The development basically follows Chapter 3 of [1]. We also outline the proof of equivalence of the new presentation with the original one in terms of their ambient categories. Thus the new construction is really another way of presenting the same structure as the original one, though from a different angle.

Some notational conventions are given in the following.

1.1 Notation.

(1) A *permutation group G on X* is a subgroup of the group of (all) permutations over X. Then $\langle G\rangle$ denote the underlying set X. For convenience, the empty set $\emptyset$ is considered as a permutation group.

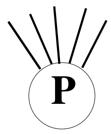
(2) Given permutation groups G and G', we write $G \cong G'$ when there is a bijection θ from $\langle G \rangle$ to $\langle G' \rangle$ such that $G' = \theta G \theta^{-1}$. Thus $\cong$ is an isomorphism between permutation groups seen as group actions. We easily know $\cong$ is an equivalence over permutation groups. We write $\cong(G, G')$ for the set of such bijections (if any) from G to G'. Again for convenience we set $\emptyset \cong \emptyset$.

(3) Given a relation R, we denote its image in the codomain by $\mathtt{im}(R)$ and the image of R^{-1} in the domain of R by $\mathtt{pre}(R)$. We also write $R \restriction A'$ for the restriction of R to A' at its domain, when A' is a subset of the domain of R.

Acknoweldgemensts. The author thanks Cliff Jones and Nobuko Yoshida for discussions and comments. This work is supported by EPSRC fellowships.

2 Group Presentation and Equivalence Theorem

Looking upon a process as a geometric object with symmetries on vertices on the top (see below), if we label them with some symbols, the order of assigning symbols is arbitrary: thus if we label the vertices in the picture below by "h_1, h_2, h_3, h_4, h_5" from the left, then any permutation of the same set (or even a different set, for that matter) of labels will do, so e.g. "h_3, h_2, h_1, h_4, h_5" from the left, is also fine, as far as we give the same structure of symmetries to them. This is the intuition underlying the following definition.



2.1 Definition. A *group presentation of a process structure*, or just a *group presentation*, is a tuple $\langle \mathbf{P},\ \mathcal{G} \rangle$, where $\mathbf{P}$ is a set of processes and $\mathcal{G}$ is a function which assigns, to each p, a set of finite permutation groups $\mathcal{G}(p) = \{G, G', \ldots\}$ such that

(i) $G, G' \in \mathcal{G}(p) \quad \Rightarrow \quad G' \cong G.$ (cf. Notation 1.1)

(ii) $G \in \mathcal{G}(p) \ \wedge\ \theta$ bijection on $\langle G \rangle \quad \Rightarrow \quad \exists G' \in \mathcal{G}(p).\ \theta \in \cong(G, G').$

$G \in \mathcal{G}(p)$ is called a *group* of p.

We sometimes write just $\mathbf{P}$ etc. to denote a group presentation. Then $\mathcal{G}(p)$ denotes the set of groups of p.

2.2 Examples. (group presentation)

(i) Any process structure[1] induces its group presentation, called its *basic group presentation*, in the following way: given a process structure $\langle \mathbf{P},\ \mathcal{H},\ \mathcal{S}\rangle$, define $\langle \mathbf{P},\ \mathcal{G}\rangle$ by taking:

$$\mathcal{G}(p) \stackrel{\text{def}}{=} \{G \mid G = \theta^{-1}\mathcal{S}(p)\theta \quad (\text{for each bijection } \theta \text{ on } \mathcal{H}(p))\}.$$

Then $\langle \mathbf{P},\ \mathcal{G}\rangle$ is a group presentation. Note for any $G, G' \in \mathcal{G}(p)$ we have $\langle G\rangle = \langle G'\rangle$ here.

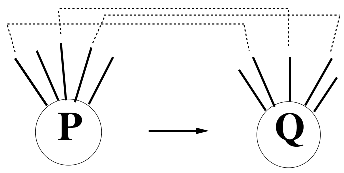
(ii) We can also regard a rooted process structure [1] as a group presentation. Take its orbits as processes and, for each orbit i, let $\{sym(P) | P \in i\}$ be its groups.

(iii) Given a group presentation $\langle \mathbf{P},\ \mathcal{G}\rangle$, we define $\langle \mathbf{P},\ \overline{\mathcal{G}}\rangle$ as follows:

$$G \in \overline{\mathcal{G}}(p) \quad \stackrel{\text{def}}{\Leftrightarrow} \quad \langle G\rangle = \underline{n} \ \text{ for } \ \sharp(\langle G\rangle) = n \ \wedge \ G \cong G' \in \mathcal{G}(p)$$

where $\underline{n} = \{0, 1, .., n-1\}$ and $\sharp(X)$ is the cardinality of X. Then $\langle \mathbf{P},\ \overline{\mathcal{G}}\rangle$ is a group presentation, called its *canonical presentation*.

Next we introduce the basic idea of relating two processes, called *correspondence* [1]. A correspondence specifies not only a tuple of processes but also how their handles are related by partial one-one map (see the figure below). Just as the definition of a process is given by looking at it from all possible directions, the definition of a correspondence is also given considering all possible variants with respect to symmetries (which are equivalent in geometric understanding) as a single entity. For a technical purpose, we also introduce its generalisation called *pseudo-correspondence*.



2.3 Definition. (correspondence and pseudo-correspondence)

(i) Let p_1 and p_2 be processes possibly from different process structures. Define $p \times q$ as the set of all triples of the form $\langle G_1,\ \delta,\ G_2\rangle$ where $G_1 \in \mathcal{G}(p_1)$, $G_2 \in \mathcal{G}(p_2)$, and δ is a partial injection from $\langle G_1\rangle$ to $\langle G_2\rangle$. Then a relation $\sim$ on $p \times q$ is given by:

$$\langle G_1,\ \delta,\ G_2\rangle \sim \langle G_1',\ \delta',\ G_2'\rangle \quad \stackrel{\text{def}}{\Leftrightarrow} \quad \delta' = \theta_2 \circ \delta \circ \theta_1^{-1}$$

where $\theta_1 \in\cong (G_1, G_2)$ and $\theta_2 \in\cong (G_1', G_2')$, respectively. We can readily check that $\sim$ is an equivalence relation on $p \times q$.

(ii) (pseudo-correspondence) A *pseudo-correspondence* from p_1 to p_2 is a triple $\langle p_1,\ \Delta,\ p_2\rangle$ where $\Delta \subset p \times q$ and

$$\langle G_1,\ \delta,\ G_2\rangle \in \Delta\ \wedge\ \langle G_1,\ \delta,\ G_2\rangle \sim \langle G_1',\ \delta',\ G_2'\rangle \quad \Rightarrow \quad \langle G_1',\ \delta',\ G_2'\rangle \in \Delta.$$

Then p_1 is its *source*, p_2 is its *target*, and Δ is its *relational component.*

(iii) (correspondence) A *correspondence* from p to q is a triple $\langle p,\ \Delta,\ q\rangle$ which is a pseudo-correspondence, where Δ is not empty, and for which $\langle G_1,\ \delta,\ G_2\rangle \in \Delta$ and $\langle G_1',\ \delta',\ G_2'\rangle \in \Delta$ implies $\langle G_1,\ \delta,\ G_2\rangle \sim \langle G_1',\ \delta',\ G_2'\rangle$. Again p is its *source*, q is its *target*, and Δ is its *relational component.*

As one may observe, $\sim$ in the above definition is essentially the same relation as $\sim$ over correspondences defined in [1]. Thus the notion of correspondence in the above definition absorbs the difference up to $\sim$ in the original definition (i.e. a single correspondence in the above definition corresponds to an *abstract correspondence* in [1]). We also note that the number of pseudo-correspondences (hence that of correspondences) with common source and target is always finite, since a collection of partial injections from a finite set to a finite set is always finite. Of course the relational component may well be infinite.

2.4 Definition and Proposition. (operations on pseudo-correspondences)

(i) *Given pseudo-correspondences $\langle p_1,\ \Delta,\ p_2\rangle$ and $\langle p_2,\ \Gamma,\ p_3\rangle$, their* composition*, written $\langle p_2,\ \Gamma,\ p_3\rangle \circ \langle p_1,\ \Delta,\ p_2\rangle$, is defined as a triple $\langle p_1,\ \Xi,\ p_3\rangle$ such that*

$$\langle G_1,\ \delta_b \circ \delta_a,\ G_3\rangle \in \Xi \quad \Leftrightarrow \quad \langle G_1,\ \delta_a,\ G_2\rangle \in \Delta \wedge \langle G_2,\ \delta_b,\ G_3\rangle \in \Gamma.$$

Then $\langle p_2,\ \Gamma,\ p_3\rangle \circ \langle p_1,\ \Delta,\ p_2\rangle$ is a pseudo-correspondence from p_1 to p_3.

(ii) *Given a pseudo-correspondence $\langle p_1,\ \Delta,\ p_2\rangle$, its* inverse*, written $\langle p_1,\ \Delta,\ p_2\rangle^{-1}$, is given by a triple $\langle p_2,\ \Delta',\ p_1\rangle$ where we have:*

$$\langle G_2,\ \delta,\ G_1\rangle \in \Delta' \quad \stackrel{\text{def}}{\Leftrightarrow} \quad \langle G_1,\ \delta^{-1},\ G_2\rangle \in \Delta.$$

Then $\langle p_1,\ \Delta,\ p_2\rangle^{-1}$ is a pseudo-correspondence, and we have:

$$(\langle p_1,\ \Delta,\ p_2\rangle^{-1})^{-1} = \langle p_1,\ \Delta,\ p_2\rangle.$$

Moreover, if $\langle p_1,\ \Delta,\ p_2\rangle$ is a correspondence, then $\langle p_1,\ \Delta,\ p_2\rangle^{-1}$ is also a correspondence..

PROOF:

(i) Suppose $\langle G_1,\ \delta_b \circ \delta_a,\ G_3\rangle \in \Delta$ and $\langle G_1,\ \delta_b \circ \delta_a,\ G_3\rangle \sim \langle G_1',\ \delta',\ G_3'\rangle$ where, for some $G_2 \in \mathcal{G}(p_2)$, δ_a is a partial injection from $\langle G_1\rangle$ to $\langle G_2\rangle$ and δ_b is a partial injection from $\langle G_2\rangle$ to $\langle G_3\rangle$. Set $\delta' = \psi_3 \circ \delta \circ \psi_1^{-1}$ by isomorphisms ψ_1 and ψ_3.

Then define $\delta'_a \stackrel{\mathrm{def}}{=} \delta_a \circ \psi_1^{-1}$ and $\delta'_b \stackrel{\mathrm{def}}{=} \psi_3 \circ \delta_b$. Then δ'_a and δ'_b are related to δ_a and δ_b respectively so that $\langle G_1,\ \delta_a,\ G_2\rangle \in \Delta$ and $\langle G_1,\ \delta_b,\ G_3\rangle \in \Gamma$. Therefore $\langle G_1,\ \delta'_b \circ \delta'_a,\ G_3\rangle \in \Xi$ but $\delta'_b \circ \delta'_a = \delta'$ hence done (see the diagram below).

$$\begin{tikzcd}
G_1 \arrow[r, "\delta_a"] \arrow[d, "\psi_1"'] & G_2 \arrow[r, "\delta_b"] \arrow[dr, "\delta'_b"] & G_3 \arrow[d, "\psi_2"] \\
G'_1 \arrow[ur, "\delta'_a"] \arrow[rr, "\delta'"] & & G'_3
\end{tikzcd}$$

(ii) Both come from the fact that $\langle G_1,\ \delta,\ G_2\rangle \sim \langle G'_1,\ \delta',\ G'_2\rangle$ if and only if $\langle G_2,\ \delta^{-1},\ G_1\rangle \sim \langle G'_2,\ \delta'^{-1},\ G'_1\rangle$. ■

Property (i) of pseudo-correspondences does not hold for the present notion of correspondences in general (cf. Discussions 3.6 [1]). This is why we introduced the former.

Now given a pseudo-correspondence $\langle p,\ \Delta,\ q\rangle$ define:

$$\langle p,\ \Delta,\ q\rangle^\circ \ \stackrel{\mathrm{def}}{=}\ \{\langle p,\ [\langle G_1,\ \delta,\ G_2\rangle]_\sim,\ q\rangle \mid \langle G_1,\ \delta,\ G_2\rangle \in \Delta\}.$$

I.e. we take the partition of Δ by $\sim$. Thus the result is a set of correspondences. We extend the map to send a set of pseudo-correspondences $\mathfrak{R}$ to a set of as follows.

$$\mathfrak{R}^\circ \ \stackrel{\mathrm{def}}{=}\ \bigcup\{\langle p,\ \Delta,\ q\rangle^\circ \mid \langle p,\ \Delta,\ q\rangle \in \mathfrak{R}\}.$$

Similarly, given a set of correspondences which have the common source and target, say $\{\langle p,\ \Delta_i,\ q\rangle\}_{i\in I}$, we define:

$$\{\langle p,\ \Delta_i,\ q\rangle\}^\bullet_{i\in I} \ \stackrel{\mathrm{def}}{=}\ \langle p,\ \cup_{i\in I}\Delta_i,\ q\rangle$$

which merges these correspondences into a single pseudo-correspondence. We extend the map to take any set of correspondences $\mathfrak{R}$ in the following way: Given such $\mathfrak{R}$, let us write $\mathtt{pre}(\mathfrak{R})$ and $\mathtt{im}(\mathfrak{R})$ for the set of source and target processes of those correspondences in $\mathfrak{R}$. Then we write $\mathfrak{R}^p_q$, where $p \in \mathtt{pre}(\mathfrak{R})$ and $q \in \mathtt{im}(\mathfrak{R})$, for a maximal subset of $\mathfrak{R}$ which only contains correspondences with the source p and q. Using the notation we define:

$$\mathfrak{R}^\bullet \ \stackrel{\mathrm{def}}{=}\ \bigcup\{\{\mathfrak{R}^p_q \mid p \in \mathtt{pre}(\mathfrak{R}),\ \ q \in \mathtt{im}(\mathfrak{R})\}^\bullet\}.$$

Then this always maps a set of correspondences to a set of pseudo-correspondences. Also one easily sees that, if $\mathfrak{R}$ is a set of pseudo-correspondences, then we have:

$$(\mathfrak{R}^\circ)^\bullet = \mathfrak{R},$$

and that, if $\mathfrak{R}$ is a set of correspondences, we have:

$$(\mathfrak{R}^\bullet)^\circ = \mathfrak{R},$$

i.e. two mappings are inverse to each other.

2.5 Definition. (p-relations)

(i) Given two group presentations $\mathbf{P}_1$ and $\mathbf{P}_2$, their *product*, written $\mathbf{P}_1 \times \mathbf{P}_2$, is the set of all correspondences from processes in $\mathbf{P}_1$ to processes in $\mathbf{P}_2$.

(ii) A *p-relation* from $\mathbf{P}_1$ to $\mathbf{P}_2$ is any subset of $\mathbf{P}_1 \times \mathbf{P}_2$. We call $\mathbf{P}_1$ its *domain* and $\mathbf{P}_2$ its *co-domain.* $\mathfrak{R}, \mathfrak{R}', \ldots$ range over p-relations. We write $p\mathfrak{R}_{\Delta}q$ for $\langle p,\ \Delta,\ q\rangle \in \mathfrak{R}$. We also write $(p : G)\mathfrak{R}_{\delta}(q : G')$ when for some Δ we have $p\,\mathfrak{R}_{\Delta}q$ and $\langle G,\ \delta,\ G'\rangle \in \Delta$.

(iii) If $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are p-relations with common domain and co-domain, then clearly their set-theoretic union, intersection, and difference are p-relations with the same domain/codomain, which we just write $\mathfrak{R}_1 \cup \mathfrak{R}_2$, $\mathfrak{R}_1 \cap \mathfrak{R}_2$, and $\mathfrak{R}_1 \backslash \mathfrak{R}_2$. Similarly if $\mathfrak{R}$ is a p-relation from $\mathbf{P}$ to $\mathbf{Q}$ then the *inverse* of $\mathfrak{R}$, written $\mathfrak{R}^{-1}$ is a p-relation from $\mathbf{Q}$ to $\mathbf{P}$ defined by:

$$\mathfrak{R}^{-1} \stackrel{\text{def}}{=} \{\langle p,\ \Delta,\ q\rangle^{-1} \mid \langle p,\ \Delta,\ q\rangle \in \mathfrak{R}\}.$$

Finally if the codomain of $\mathfrak{R}_1$ and the domain of $\mathfrak{R}_2$ coincide, we define:

$$\mathfrak{R}_2 \circ \mathfrak{R}_1 \stackrel{\text{def}}{=} (\mathfrak{R}_2{}^{\bullet} \circ \mathfrak{R}_1{}^{\bullet})^{\circ}.$$

which is called the *composition* of $\mathfrak{R}_1$ and $\mathfrak{R}_2$ (in this order).

Observe $\mathfrak{R}_2 \circ \mathfrak{R}_1$ above is well-defined because pseudo-correspondences are composable. Also note that we have defined a p-relation without using the condition on equivalence over correspondences, cf. Definition 3.5 of [1]. This is because an equivalence class of p-relation by $\sim$ in the original construction is represented by a *single* correspondence in the present setting.

With Definition 2.5, a p-relation is already a process structure.

2.6 Proposition *Given a p-relation $\mathfrak{R}$, the following data define a group presentation.*

(i) The set of processes: $\mathfrak{R}$ itself (cf. 3.10 (i) of [1]).

(ii) Groups: $\mathcal{G}(\langle p,\ \Delta,\ q\rangle) \stackrel{\text{def}}{=} \{\phi(\langle G,\ \delta,\ G'\rangle) \mid \langle G,\ \delta,\ G'\rangle \in \Delta\}$ where we define $\phi(\langle G,\ \delta,\ G'\rangle)$ as a permutation group over a set $(\langle G\rangle \backslash \mathtt{pre}(\delta)) \uplus \delta \uplus (\langle G'\rangle \backslash \mathtt{im}(\delta))$ induced by the following group action: we take $\{\langle g,\ g'\rangle \mid g \in G \wedge g' \in G' \wedge \delta \circ g = g' \circ \delta\}$ as a group, multiplication and the inverse defined componentwise, and let the group elements act on $(\langle G\rangle \backslash \mathtt{pre}(\delta)) \uplus \delta \uplus (\langle G'\rangle \backslash \mathtt{im}(\delta))$ as follows.

$$\begin{array}{rcl} \langle g,\ g'\rangle(x) & = & g(x) \quad (x \in \langle G\rangle \backslash \mathtt{pre}(\delta)) \\ \langle g,\ g'\rangle(x \mapsto y) & = & g(x) \mapsto g'(y). \\ \langle g,\ g'\rangle(y) & = & g'(y) \quad (y \in \langle G'\rangle \backslash \mathtt{im}(\delta)). \end{array}$$

PROOF: This is because $y = \delta(x)$ implies $g'(y) = g \circ \delta(x)$ by definition, from which we deduce that $\langle g,\ g' \rangle$ as defined does give a bijection on $(\langle G \rangle \backslash \mathtt{pre}(\delta)) \uplus \delta \uplus (\langle G' \rangle \backslash \mathtt{im}(\delta))$. The rest is easy. ■

We proceed to the notion which is an analogue of functions over sets, called *p-maps*.

2.7 Definition. (p-map and isomorphism)

(i) A correspondence $\langle p,\ \Delta,\ q \rangle$ is *surjective* if any δ in $\langle G,\ \delta,\ G' \rangle \in \Delta$ is surjective. It is *symmetry preserving* when, if $\langle G,\ \delta,\ G' \rangle$ implies for any $g \in G$ there exists $g' \in G'$ such that $\delta \circ g = g' \circ \delta$. Then a p-relation $\mathfrak{R}$ from $\mathbf{P}$ to $\mathbf{Q}$ is a *p-map* when

   (1) For each $p \in \mathbf{p}$ there is a unique $q \in Q$ and a unique Δ such that $\langle p,\ \Delta,\ q \rangle \in \mathfrak{R}$.

   (2) $\langle p,\ \Delta,\ q \rangle \in \mathfrak{R}$ implies Δ is symmetry-preserving and surjective.

(ii) A p-map whose inverse is also a p-map is called an *isomorphism*. By the *image* of a p-map $\mathfrak{R}$, we mean a sub process structure of $\mathtt{cod}(\mathfrak{R})$ given by $\{q \mid \exists p.\ p\mathfrak{R}_\delta q\}$. An *isomorphism* is a p-map whose inverse is also a p-map. Given $\mathbf{P}$, the *identity on* $\mathbf{P}$, written $\mathbf{ID_P}$, is the set of correspondences of a form $\langle p,\ \Delta,\ p \rangle$ for each $p \in \mathbf{P}$ where $\Delta \stackrel{\text{def}}{=} \{\langle G,\ \delta,\ G' \rangle \mid G, G' \in \mathcal{G}(p)\}$ in which δ should be a bijection from $\langle G \rangle$ to $\langle G' \rangle$ inducing an isomorphism from G to G'.

Note the "uniqueness" of a p-map in (i) above is *real* set-theoretic uniqueness, not modulo some equivalence, cf. Definition 3.7 [1].

2.8 Examples. (p-map and isomorphism)

(i) Assume given $\mathbf{P}_1$ and $\mathbf{P}_2$. Regarding $\mathbf{P}_1 \times \mathbf{P}_2$ as a process structure by Proposition 2.6, define two p-maps $\Pi_1 : \mathbf{P}_1 \times \mathbf{P}_2 \to \mathbf{P}_1$ and $\Pi_2 : \mathbf{P}_1 \times \mathbf{P}_2 \to \mathbf{P}_2$ such that:

   (1) $\langle\langle p_1,\ \Delta,\ p_2 \rangle,\ \Gamma,\ p_1 \rangle \in \Pi_1$ if and only if the following holds: for each $\langle G_1,\ \delta,\ G_2 \rangle \in \Delta$, $\langle\langle G_1,\ \delta,\ G_2 \rangle,\ \pi,\ G_1' \rangle \in \Gamma$ iff, for some $\theta \in \cong (G_1, G_1')$, (a) $\pi : (x_1 \mapsto x_2) \mapsto \theta(x_1)$ (for $(x_1 \mapsto x_2) \in \delta$), (b) $\pi : x_1 \mapsto \theta(x_1)$ (for $x_1 \in \langle G_1 \rangle \backslash \mathtt{pre}(\delta)$), (c) else undefined.

   (2) $\langle\langle p_1,\ \Delta,\ p_2 \rangle,\ \Gamma,\ p_2 \rangle \in \Pi_2$ if and only if the following holds: for each $\langle G_1,\ \delta,\ G_2 \rangle \in \Delta$, $\langle\langle G_1,\ \delta,\ G_2 \rangle \in \Delta,\ \pi,\ G_2' \rangle \in \Gamma$ iff, for some $\theta \in \cong (G_2, G_2')$, (a) $\pi : (x_1 \mapsto x_2) \mapsto \theta(x_2)$ (for $(x_1 \mapsto x_2) \in \delta$), (b) $\pi : x_2 \mapsto \theta(x_2)$ (for $x_2 \in \langle G_2 \rangle \backslash \mathtt{im}(\delta)$), (c) else undefined.

   Then Π_1 and Π_2 are p-maps, called *projection p-maps* of $\Pi_1 : \mathbf{P}_1 \times \mathbf{P}_2$. We can readily check that these projections satisfy the familiar universality property.

(ii) Given a group presentation $\langle \mathbf{P},\ \mathcal{G} \rangle$, we construct a p-relation from $\langle \mathbf{P},\ \mathcal{G} \rangle$ to its canonical presentation $\langle \mathbf{P},\ \overline{\mathcal{G}} \rangle$ (defined in Example 2.2 (iii)) as follows: $\mathfrak{R}$ contains $\langle p,\ \Delta,\ p \rangle \in \mathfrak{R}$ for each $p \in \mathbf{P}$, which collectively covers $\mathfrak{R}$, and for which we have:

$$\langle G,\ \theta,\ \overline{G} \rangle \in \Delta \quad \Leftrightarrow \quad \theta \in \cong (G, \overline{G})$$

where we write the (unique) element of $\mathcal{G}(p)$ as $\overline{G}$, for which we have $\overline{G} \cong G$ by definition. Then $\mathfrak{R}$ is clearly an isomorphism. Thus any canonical presentation of a group presentation is isomorphic to the original presentation.

2.9 Proposition.

(i) *Given two p-maps composable as p-relations, their composition is again a p-map.*

(ii) *If $\mathfrak{R}$ is from $\mathbf{P}$ to $\mathbf{Q}$ then $\mathfrak{R} \circ \mathbf{ID_P} = \mathbf{ID_Q} \circ \mathfrak{R} = \mathfrak{R}$. Also if $\mathfrak{R}_1$ from $\mathbf{P}$ to $\mathbf{Q}$ and $\mathfrak{R}_2$ from $\mathbf{Q}$ to $\mathbf{P}$ are isomorphisms, we have $\mathfrak{R}^{-1} \circ \mathfrak{R} = \mathbf{ID_P}$ and $\mathfrak{R} \circ \mathfrak{R}^{-1} = \mathbf{ID_Q}$.*

(iii) *We say two group presentations are* isomorphic *if there is an isomorphism between them. Then $\mathbf{P}$ and $\mathbf{Q}$ are isomorphic if and only if there is a bijection between $\mathbf{P}$ and $\mathbf{Q}$ such that for each pair of related process, say p and q, for some $G \in \mathcal{G}(p)$ and $G' \in \mathcal{G}(q)$, we have $G \cong G'$.*

Proof: (i) is just as the proof of Proposition 3.8 of [1] while (ii) is easy. (iii) is again just as Proposition 3.9 of [1]. ■

The following definition is a little stronger than what was given in [1] in that we require that the reciprocation operation (corresponding to the relational inverse) for an isomorphism coincides with its categorical inverse. This becomes useful for the proof of the equivalence theorem later.

2.10 Definition. A *complete boolean allegory* is a category in which, for each pair of objects A and B, we have:

(i) $hom(A, B)$ forms a complete boolean algebra, with operations $\bigcup, \bigcap, (\,\cdot\,)^{\mathcal{C}}$, the first two of which are compatible with categorical composition $\circ$, i.e. $(\bigcup_i R_i) \circ S = \bigcup_i (R_i \circ S)$ and $S \circ (\bigcup_i R_i) = \bigcup_i (S \circ R_i)$, similarly for $\bigcap$.

(ii) There is an operation $(\,\cdot\,)^{-1}$ called *reciprocation*, which sends $hom(A, B)$ to $hom(B, A)$ so that $(\mathfrak{R}^{-1})^{-1} = \mathfrak{R}$, and for which we have:

$$(R_1 \circ R_2)^{-1} = R_2^{-1} \circ R_1^{-1} \qquad (\textstyle\bigcup_i R_i)^{-1} = \bigcup_i (R_i^{-1}) \qquad (R^{\mathcal{C}})^{-1} = (R^{-1})^{\mathcal{C}}$$

as well as, if R is an isomorphism, then R^{-1} coincides with the inverse of R.

Functors between two complete boolean allegories are those between the underlying categories which also preserve these operations. Two complete boolean allegories are *equivalent as allegories* when they are categorically equivalent involving functors in the above sense.

2.11 Proposition. (PS^g_{rel} and PS^g) *The collection of group presentations and p-relations between them form a complete boolean allegory. This allegory (as a category) contains the category of the same set and p-maps as its subcategory. We call these two* *structures,* PS^g_{rel} *and* PS^g*, respectively. Note isomorphisms coincide in* PS^g_{rel} *and* PS^g*.*

PROOF: Easy by observing, for example, that $\mathfrak{R}^{-1}$ is, in our case, a (p-)relational inverse, so that, if $\mathfrak{R}$ is an isomorphism, $\mathfrak{R}^{-1}$ precisely coincides with the categorical inverse. ∎

We note that PS_{rel} in [1] is a complete boolean allegory with the above stronger definition, as will be readily checked. The following theorem establishes the formal correspondence between the universes of the original presentation and those of the group presentation. The proof sketch of the theorem is given at the end of the section.

2.12 Theorem. PS_{rel} *of [1] and* PS^g_{rel} *are equivalent as allegories, where involved functors induce categorical equivalence between* PS *of [1] and* PS^g*.*

We now move to the quotient construction. One of the neatest points of the group presentation is that we can talk of "the" quotient of an equivalence p-relation, without using a choice function, cf. 3.13 of [1].

2.13 Definition. (equivalence p-relation and equivalence classes)

- (i) A p-relation $\mathfrak{R}$ on $\mathbf{P}$ (i.e. from $\mathbf{P}$ to $\mathbf{P}$) is an *equivalence p-relation* if $\mathbf{ID_P} \subset \mathfrak{R}$, $\mathfrak{R} \circ \mathfrak{R} \subset \mathfrak{R}$, and $\mathfrak{R}^{-1} = \mathfrak{R}$.
- (ii) Given an equivalence p-relation $\mathfrak{R}$ on $\mathbf{P}$, the *equivalence class* of a process $p \in \mathbf{P}$ with respect to $\mathfrak{R}$ is defined as: $[p]_{\mathfrak{R}} \stackrel{\text{def}}{=} \{q \in \mathbf{P} \mid \ p\,\mathfrak{R}_{\Delta}\, q\}$.
- (iii) Given an equivalence p-relation $\mathfrak{R}$ on $\mathbf{P}$, let $G \in \mathcal{G}(p)$ for $p \in \mathbf{P}$. Then the $\mathfrak{R}$-*effective subset of* $\langle G\rangle$ is given by:

$$\langle G\rangle|\mathfrak{R} \stackrel{\text{def}}{=} \{x | x \in \bigcap \mathtt{im}(\delta) \ \text{ for } \delta \text{ s.t. } (p:G)\mathfrak{R}_{\delta}(p:G).\}$$

2.14 Proposition. *Let* $G \in \mathcal{G}(p)$ *in the following.*

- (i) $x \in \langle G\rangle|\mathfrak{R}$ *iff* $x \in \mathtt{im}(\delta)$ *for every* δ *such that* $\langle G',\ \delta,\ G\rangle \in \Delta$ *and* $q\,\mathfrak{R}_{\Delta} p$ *for every* q *and* Δ *iff* $x \in \mathtt{pre}(\delta)$ *for every* δ *such that* $\langle G,\ \delta,\ G'\rangle \in \Delta$ *and* $q\,\mathfrak{R}_{\Delta} p$ *for every* q *and* Δ*.*
- (ii) $\{\delta \restriction (\langle G\rangle|\mathfrak{R}) \mid (p:G)\mathfrak{R}_{\delta}(p:G)\}$ *forms a permutation group over* $\langle G\rangle|\mathfrak{R}$*, which we denote by* $sym(\langle G\rangle|\mathfrak{R})$*. If* $p\,\mathfrak{R}_{\Delta} q$ *and* $G' \in \mathcal{G}(q)$*, then* $\langle G\rangle|\mathfrak{R} \cong \langle G'\rangle|\mathfrak{R}$*.*
- (iii) *Let* $G|\mathfrak{R} \stackrel{\text{def}}{=} \{g \restriction (\langle G\rangle|\mathfrak{R}) \mid g \in G\}$*. Then* $G|\mathfrak{R}$ *is a subgroup of* $sym(\langle G\rangle|\mathfrak{R})$*.*

PROOF:

- (i) The first "iff" holds by transitivity of $\mathfrak{R}$ while the second "iff" holds from symmetry of $\mathfrak{R}$.

(ii) By reflexivity of $\mathfrak{R}$, it surely contains $\mathbf{id}(\langle G\rangle) \restriction (\langle G\rangle|\mathfrak{R})$, i.e. $\mathbf{id}(\langle G\rangle|\mathfrak{R})$. Now let $\delta = \delta' \restriction (\langle G\rangle|\mathfrak{R})$, then, by the second iff of (i) above, $\mathtt{im}(\delta) = \langle G\rangle|\mathfrak{R}$ too. Thus its inverse is given by ${\delta'}^{-1} \restriction (\langle G\rangle|\mathfrak{R})$, which is also in $sym(\langle G\rangle|\mathfrak{R})$. Finally we have $(\delta_2 \restriction (\langle G\rangle|\mathfrak{R})) \circ (\delta_1 \restriction (\langle G\rangle|\mathfrak{R})) = (\delta_2 \circ \delta_1 \restriction (\langle G\rangle|\mathfrak{R}))$ by (i) above which is in $sym(\langle G\rangle|\mathfrak{R})$ again.

(iii) Let $g \in G$. Then by reflexivity and the definition of $\mathbf{ID_P}$ in Definition 2.5 (ii), we know $(p : G)\mathfrak{R}_g(p : G)$. Hence $g \restriction (\langle G\rangle|\mathfrak{R}) \in sym(\langle G\rangle|\mathfrak{R})$. With the same reasoning as (ii) above, we know such elements form a group, hence done. ∎

2.15 Proposition. (quotient) *Given an equivalence p-relation $\mathfrak{R}$ over $\mathbf{P}$, the* quotient of $\mathbf{P}$ by $\mathfrak{R}$, *written $\mathbf{P}/\mathfrak{R}$, is given by the following data.*

(i) *Processes:* $\{[p]_{\mathfrak{R}} \mid p \in \mathbf{P}\}$.

(ii) *Groups:* $\mathcal{G}([p]_{\mathfrak{R}}) \stackrel{\text{def}}{=} \{sym(\langle G\rangle|\mathfrak{R}) \mid G \in \mathcal{G}(q) \text{ for } q \in [p]_{\mathfrak{R}}\}$.

Then $\mathbf{P}/\mathfrak{R}$ is a group presentation.

Proof: By Proposition 2.14 (ii) we know that if $G, G' \in \mathcal{G}([p]_{\mathfrak{R}})$ then $\langle G\rangle = \langle G'\rangle$ and $G \simeq G'$. So suppose there is a bijection θ on $\langle G\rangle$ for $G \in \mathcal{G}([p]_{\mathfrak{R}})$, i.e. on $\langle G'\rangle|\mathfrak{R}$ for some $G' \in \mathcal{G}(q)$ for $q \in [p]_{\mathfrak{R}}$. Such a bijection is included at least in one bijection on $\langle G'\rangle$, say θ'. By definition $\theta' G' {\theta'}^{-1} \in \mathcal{G}(q)$. Then take $\langle \theta' G' {\theta'}^{-1}\rangle|\mathfrak{R}$ which gives a desired isomorphic group. ∎

Thus quotienting is just done by taking the equivalence classes and transforming all groups owned by processes in each equivalence class using the equivalence p-relation. This is much cleaner than the construction in 3.13 of [1] where we need to use a choice function, though the latter surely serves its purpose, so that technically it does not induce any problem. We also note that Proposition 2.14 (iii) tells us that, while the set underlying each permutation group (corresponding to handles) will be restricted, the symmetry itself increases, cf. discussions after 2.18 and 3.15 of [1]. Such a fact is also more clearly seen in the present context. Also note that, as in [1], multiple equivalences give rise to the same quotient, since an equivalence can have redundant correspondences which are to be cancelled by restricting the domain. The situation is quite different from the quotient of a set.

We next establish the relationship between p-maps and quotient constructions, then move to the proof of the main theorem.

2.16 Proposition.

(i) *Given an equivalence $\mathfrak{R}$ over $\mathbf{P}$, a* natural map $\eta_{\mathfrak{R}}$ *is the set of correspondences from $\mathbf{P}$ to $\mathbf{P}/\mathfrak{R}$ such that, for each $q \in [p]_{\mathfrak{R}}$, we have a unique correspondence $\langle q,\ \Delta,\ [p]_{\mathfrak{R}}\rangle \in \eta_{\mathfrak{R}}$ such that, with $G' \in \mathcal{G}(r)$ and $r \in [p]_{\mathfrak{R}}$,*

$$\langle G,\ \delta,\ sym(\langle G'\rangle)\rangle \in \Delta \quad \Leftrightarrow \quad (q : G)\,\mathfrak{R}_\delta\,(r : G') \ \wedge\ \mathtt{im}(\delta) = \langle G'\rangle|\mathfrak{R}$$

Then $\eta_{\mathfrak{R}}$ is a p-map surjective on processes.

(ii) *Suppose a p-map $\mathfrak{R} : \mathbf{P} \to \mathbf{Q}$ covers all processes in $\mathbf{Q}$. Define a set of correspondences $\sim'$ over $\mathbf{P}$ by putting $(p : G) \sim'_{\delta} (p' : G')$ if and only if, for some $q \in \mathbf{Q}$, we have:*

$$(p : G_1)\mathfrak{R}_{\pi_1}(q : G) \ \wedge \ (p' : G_2)\mathfrak{R}_{\pi_2}(q : G) \ \wedge \ \delta = \pi_2^{-1} \circ \pi_1 \ .$$

Then $\sim = \sim' \cup \mathbf{ID_P}$ is an equivalence p-relation. Moreover, there is an isomorphism $\mathfrak{R}'$ from $\mathbf{P}/\sim$ to $\mathbf{Q}$ such that $\mathfrak{R} = \mathfrak{R}' \circ \eta_{\sim}$.

PROOF:

(i) To show $\eta_{\mathfrak{R}}$ is a p-map, we should verify that its elements are correspondences which are uniquely defined for each process in the source and that each correspondence is surjective and symmetry preserving. If so, then that it is surjective on $\mathbf{P}/\mathfrak{R}$ is obvious. Suppose $\langle q,\ \Delta,\ [p]_{\mathfrak{R}}\rangle \in \eta_{\mathfrak{R}}$. We first show this triple is a pseudo-correspondence. Put $\langle G_1,\ \delta,\ G_2\rangle \in \Delta$, $\theta_1 \in \cong (G_1, G_1')$ and $\theta_2 \in \cong (G_2, G_2')$. But $\theta_2 \in \cong (G_2, G_2')$ means for some $r, r' \in [p]_{\mathfrak{R}}$ we have $(r : \underline{G_2})\,\mathfrak{R}_{\delta}\,(r' : \underline{G_2'})$ with $\underline{G_2}|\mathfrak{R} = G_2$ and $\underline{G_2'}|\mathfrak{R} = G_2'$. Then by transitivity on $\mathfrak{R}$ we get $(q : G_1')\,\mathfrak{R}_{\theta_2\circ\delta\circ\delta^{-1}}\,(r' : \underline{G_2'})$ which shows $\langle G_1',\ \theta_2 \circ \delta \circ \delta^{-1},\ G_2'\rangle \in \Delta$, as required. Second we show it is a correspondence. If $\langle G_1,\ \delta,\ G_2\rangle \in \Delta$ and $\langle G_1',\ \delta',\ G_2'\rangle \in \Delta$, then we can write, understanding $\underline{G_2}$ etc. as above, $(q : G_1)\,\mathfrak{R}_{\delta}\,(r : \underline{G_2})$ and $(q : G_1')\,\mathfrak{R}_{\delta'}\,(r' : \underline{G_2'})$ for $r, r' \in [p]_{\mathfrak{R}}$. Take $\theta_1 \in \cong (G_1, G_1')$ and let

$$\delta_2 \stackrel{\text{def}}{=} \delta' \circ \theta_1 \circ \delta^{-1}.$$

By transitivity we have $(r : G_2)\mathfrak{R}_{\theta_2}(r : G_2)$. Together with the construction of θ_2, this implies $\theta_2 \in \cong (G_2, G_2')$, from which we have $\delta' = \theta_2 \circ \delta \circ \theta_1^{-1}$, as required. For surjectivity, suppose $\langle G_1,\ \delta,\ G_2\rangle \in \Delta$. Then δ is surjective by Proposition 2.15 (i) (the first iff) and by $G_2 = \underline{G_2}|\mathfrak{R}$ for some $\underline{G_2}$. Finally for symmetry preservation, we again rewrite $\langle G_1,\ \delta,\ G_2\rangle$ as $(q : G_1)\,\mathfrak{R}_{\delta}\,(u : \underline{G_2})$ and let $g_1 \in G_1$. Noting $(q : G_1)\,\mathfrak{R}_{\delta\circ g_1}\,(q : G_1)$, we put

$$g_2 \stackrel{\text{def}}{=} \delta \circ g_1 \circ \delta^{-1}.$$

Then g_2 is a partial function from $\langle \underline{G_2}\rangle|\mathfrak{R}$ to $\langle \underline{G_2}\rangle|\mathfrak{R}$, i.e. in $sym(\langle \underline{G_2}\rangle|\mathfrak{R}) = G_2$. But by definition we get $g_2 \circ \delta = \delta \circ g_1$, hence done.

(ii) Clearly $\sim'$ is symmetric and transitive. Adding $\mathbf{ID_P}$ does not change these properties so that $\sim$ is an equivalence. Now we construct the isomorphism from $\mathbf{P}/\sim$ to $\mathbf{Q}$. It is quite simple. Let $p\mathfrak{R}_{\Delta}q$. Then define:

$$\Delta|\sim \ \stackrel{\text{def}}{=} \ \{\langle sym(\langle G\rangle|\sim),\ \delta \restriction (\langle G\rangle|\sim),\ G'\rangle \mid G\Delta_{\delta}G'\}$$

Now the isomorphism is defined by:

$$\Psi \ \stackrel{\text{def}}{=} \ \{\langle [p]_{\sim},\ \Delta|\sim,\ q\rangle \mid p'\mathfrak{R}_{\Delta}q \ \wedge \ p' \in [p]_{\sim}\}$$

which is easily a p-map (symmetry preservation comes from that of the underlying correspondences in $\mathfrak{R}$). By construction it is bijective on processes and induce bijection on handles. Thus if the symmetry preservation in the reverse direction holds we are done. Let $([p]_\sim : G_p)\Psi_\delta(q : G_q)$ and $g \in G_q$. Take $g' \stackrel{\text{def}}{=} \delta^{-1} \circ g \circ \delta$. By $g \in G_q$ we have $G_p \Delta_{\delta \circ g} G_q$ so that, by definition, $([p]_\sim : G_p) \sim_{g'} ([p]_\sim : G_p)$ which shows, by definition of quotient, $g \in G_p$. But we have

$$\delta \circ g' = g \circ \delta$$

hence done. ■

2.17 Proof of Theorem 2.12. We conclude the paper with the outline of the proof of Theorem 2.12. As has been observed, the correspondence between the group presentation and the original one is quite close, so that the proof poses no difficult problem. We use the following characterisation of allegorical equivalence, which comes from a similar fact in category theory (cf. IV: 4 of [2]).

Proposition. *Complete boolean allegories A and B are equivalent if and only if there is a functor (as an allegory) Ψ from A to B such that:*

(i) Ψ *is faithful and full,*

(ii) *For any $b \in B$ there is $\Psi(a)$ such that $b \simeq \Psi(a)$, and*

(iii) Ψ *induces an isomorphism between $hom(a_1, a_2)$ of A and $hom(\Psi(a_1), \Psi(a_2))$ in B with respect to operations $\cap, \cup, (\cdot)^C, (\cdot)^{-1}$.*

Proof: Standard (cf. *op.cit.*) except for the following point: to prove the "if" direction, we construct, as in *op.cit.*, an arrow which is to become a natural isomorphism for $\Psi \circ \Phi$, where Φ is what is to become a right adjoint to Ψ, by taking, for each $b \in B$, some isomorphic $\Psi(a)$ (which exists by the above definition) and set $a = \Phi(b)$. Then we can show $\simeq: b \to \Psi(a)$ is universal to Ψ, hence Φ defines a *categorical* right adjoint to Ψ. Thus we get the following diagram:

$$\begin{tikzcd} b \arrow[r, "\simeq"] \arrow[d, "f"'] & \Psi(a) \arrow[d, "\Psi \circ \Phi(f)"] & a \arrow[d, "\Phi(f)"] \\ b' \arrow[r, "\simeq"] & \Psi(a') & a' \end{tikzcd}$$

Next we have to show that Φ is also an *allegorical* functor, i.e. that it preserves the operations of complete boolean algebra. Since Ψ induces boolean isomorphisms between related homsets, we only have to check that, on the left hand side of the above diagram, $\Psi \circ \Phi$ gives a boolean isomorphism. Since universal arrows are isomorphisms, this is checked just by showing $\Psi \circ \Phi$ gives a boolean homomorphism. We only show the cases of $\cup$ and $(\cdot)^{-1}$. Other cases are similar. Let i and i' be the isomorphisms from b to $\Psi(a)$ and from b' and $\Psi(a')$ in the above diagram, respectively.

$$\begin{aligned}\Psi \circ \Phi(\bigcup f) &= i' \circ \bigcup f \circ i^{-1} \\ &= \bigcup (i' \circ f \circ i^{-1}) \\ &= \bigcup (\Psi \circ \Phi(f))\end{aligned}$$

as required. For the reciprocation, noting that, for isomorphisms, the categorical inverse and the reciprocation coincide by Definition 2.10,

$$\begin{aligned}\Psi \circ \Phi(f^{-1}) &= i \circ f^{-1} \circ i'^{-1} \\ &= (i' \circ f \circ i^{-1})^{-1} \\ &= (\Psi \circ \Phi(f))^{-1}\end{aligned}$$

hence done. ■

We construct a faithful and full functor from PS_{rel} to PS^{g}_{rel} which has the properties as specified above. The construction is very simple.

2.18 Definition. (Functor Ψ) We define the mapping Ψ as follows.

(i) For an object $\mathbf{P}$ of PS_{rel}, define $\Psi(\mathbf{P})$ as its basic group presentation (cf. Example 2.2 (i)).

(ii) For an arrow $\mathfrak{R}$ of PS_{rel}, define $\Psi(\mathfrak{R})$ as a set $\{\langle p,\ \hat{\delta},\ q\rangle \mid \langle p,\ \delta,\ q\rangle \in \mathfrak{R}\}$ where, assuming $G_1 \in \mathcal{G}(p)$ and $G_2 \in \mathcal{G}(q)$,

$$\hat{\delta} \stackrel{\text{def}}{=} \{\langle G_1,\ \delta',\ G_2\rangle \mid \delta' = \theta_2 \circ \delta \circ \theta_1,\ \theta_1 \in \cong(\mathcal{S}(p), G_1),\ \theta_2 \in \cong(\mathcal{S}(q), G_2)\}.$$

We first observe the existence of an embedding map at the level of objects which goes in the inverse direction of Ψ.

2.19 Proposition. *Given any group presentation $\langle \mathbf{P},\ \mathcal{G}\rangle$, define the following data, choosing a single G_p from $\mathcal{G}(p)$ in $p \in \mathbf{P}$.*

(i) Processes: $\mathbf{P}$.

(ii) Handles: $\mathcal{H}(p) \stackrel{\text{def}}{=} \langle G_p\rangle$.

(iii) Symmetries: G_p *itself.*

Then the above data define a process structure. Write it $[\![\langle \mathbf{P},\ \mathcal{G}\rangle]\!]$. Then we have $\Psi([\![\langle \mathbf{P},\ \mathcal{G}\rangle]\!]) \simeq \langle \mathbf{P},\ \mathcal{G}\rangle$.

Proof: That $[\![\langle \mathbf{P},\ \mathcal{G}\rangle]\!]$ is a process structure is trivial, while the existence of an isomorphism is easily verified using Proposition 2.9 (iii). ■

Our next observation concerns how Ψ acts on "prime arrows" in PS_{rel}.

2.20 Proposition.

(i) *Let $\mathfrak{R}$ from $\mathbf{P}$ to $\mathbf{Q}$ in PS_{rel} be an* abstract correspondence *(cf. 3.6 of [1]), i.e. a p-relation such that $\langle p,\ \delta,\ q\rangle, \langle p',\ \delta',\ q'\rangle \in \mathfrak{R}$ implies $p' = p$ and $q' =$ and $\langle p,\ \delta,\ q\rangle \sim \langle p,\ \delta',\ q\rangle$. Then $\Psi(\mathfrak{R})$ is a singleton, i.e. consists of only one correspondence. Moreover different abstract correspondences result in different singletons. Finally any singleton in $hom(\Psi(\mathbf{P}), \Psi(\mathbf{Q}))$ arises in this way.*

(ii) *In PS_{rel}, any non-empty element of $hom(\mathbf{P}, \mathbf{Q})$ is a union of abstract correspondences, and, in PS^g_{rel}, any non-empty element of $hom(\Psi(\mathbf{P}), \Psi(\mathbf{Q}))$ is as a union of singletons.*

PROOF: Regarding (i), we first remember that $\langle p,\ \delta,\ q\rangle \sim \langle p,\ \delta',\ q\rangle$ is defined as $\delta = \rho_q \circ \delta \circ \rho_p$ where $\rho_p \in \mathcal{S}(p)$ and $\rho_q \in \mathcal{S}(q)$. But ρ_p and ρ_q are themselves in $\cong(\mathcal{S}(p), G_1)$ and $\cong(\mathcal{S}(q), G_2)$ of Definition 2.19 (ii) respectively. Thus, in terms of the mapping in Definition 2.18, $\hat{\delta} = \hat{\delta}$ if and only if $\delta \sim \delta'$ in $\mathfrak{R}$ of PS_{rel}, from which the the first two clauses follow. For the final clause, we explicitly construct the map in the inverse direction, from a correspondence in group presentation to an abstract correspondence, in the following way: Since the object part of Ψ clearly gives a bijection from $\mathbf{P}$ to $\Psi(\mathbf{P})$, taking any $p \in \Psi(\mathbf{P})$ we take $\Phi(p)$ as its original process in $\mathbf{P}$. Writing G_p for $\mathcal{S}(p)$, we put:

$$\Phi(\langle p,\ \Delta,\ q\rangle) \stackrel{\text{def}}{=} \{\langle p,\ \delta,\ q\rangle \mid \langle G_p,\ \delta,\ G_q\rangle \in \Delta\}.$$

By construction Φ is the inverse of Ψ, which shows Ψ gives a bijection from abstract correspondences to correspondences, hence done. Finally (ii) is an easy but important fact concerning relational universes on both sides. ■

Using the above results we establish the basic properties of Ψ.

2.21 Proposition.

(i) $\Psi(\bigcup_{i\in I} \mathfrak{R}_i) = \bigcup_{i\in I} \Psi(\mathfrak{R}_i)$ (including $I = \emptyset$).
(ii) $\Psi(\bigcap_{i\in I} \mathfrak{R}_i) = \bigcap_{i\in I} \Psi(\mathfrak{R}_i)$ (including $I = \emptyset$).
(iii) $\Psi(\mathfrak{R}^{-1}) = \Psi(\mathfrak{R})^{-1}$.
(iv) $\Psi(\mathfrak{R}^C) = \Psi(\mathfrak{R})^C$.
(v) $\Psi(\mathfrak{R}_2 \circ \mathfrak{R}_1) = \Psi(\mathfrak{R}_2) \circ \Psi(\mathfrak{R}_1)$.
(vi) $\Psi(\mathbf{ID_P}) = \mathbf{ID}_{\Psi(\mathbf{P})}$.
(vii) Ψ gives a bijection from $hom(\mathbf{P}, \mathbf{Q})$ to $hom(\Psi(\mathbf{P}), \Psi(\mathbf{Q}))$.
(viii) $\mathfrak{R}$ is a p-map if and only if $\Psi(\mathfrak{R})$ is a p-map.

PROOF: (i) is obvious from the definition. For (ii), by Proposition 2.20, we can think of Ψ as an injection from a set of abstract correspondences (in PS_{rel}) to a set of correspondences (in PS^g_{rel}). Thus, if we write $r, r', \ldots$ for abstract correspondences, then we have:

$$(1) \qquad \Psi(\bigcup_{i\in I} r_i) \;=\; \{\Psi(r_i)\}_{i\in I}.$$

In addition, Proposition 2.20 (i) tells us:

$$(2) \qquad r \neq r' \quad \Rightarrow \quad \Psi(r) \neq \Psi(r')$$

Then we have:

$$\begin{array}{rcll} \Psi(\bigcap_{i\in I}\bigcup_{j\in J_i} r_j) & = & \bigcap_{i\in I}\{\Psi(r_j)\}_{j\in J_i} & \text{(by (2) above)} \\ & = & \bigcap_{i\in I}\Psi(\bigcup_{j\in J_i}) & \text{(by (1) above)} \end{array}$$

hence done. (iii) comes from, following a notation for an abstract correspondence, $\Psi(r^{-1}) = \Psi(r)^{-1}$ together with (1). (iv) is easy by (1) and (2). For (v), we have:

$$\begin{array}{rcl} \Psi(\mathfrak{R}_2 \circ \mathfrak{R}_1) & = & \Psi(\{\langle p_1,\ \delta_2\circ\delta_1,\ p_2\rangle \mid \langle p_1,\ \delta_1,\ p'\rangle \in \mathfrak{R}_1 \ \wedge\ \langle p',\ \delta_2,\ p_2\rangle \in \mathfrak{R}_2\}) \\ & = & \bigcup\{(\langle p_1,\ \hat{\delta_2},\ p'\rangle \circ \langle p',\ \hat{\delta_2},\ p_2\rangle)^{\circ} \mid \langle p_1,\ \delta_1,\ p'\rangle \in \mathfrak{R}_1 \wedge \langle p',\ \delta_2,\ p_2\rangle \in \mathfrak{R}_2\} \\ & = & (\Psi(\mathfrak{R}_2)^{\bullet} \circ \Psi(\mathfrak{R}_1)^{\bullet})^{\circ} \\ & = & \Psi(\mathfrak{R}_2) \circ \Psi(\mathfrak{R}_1). \end{array}$$

(vi) follows from (iii) and (v). For (vii), we first note that (i) above and Proposition 2.20 ((ii) and the final clause of (i)) imply that Ψ is a surjection on $hom(\Psi(\mathbf{P}), \Psi(\mathbf{Q}))$. Suppose $\Psi(\bigcup r_i) = \Psi(\bigcup r'_j)$. By (1) we get $\{\Psi(r_i)\} = \{\Psi(r'_j)\}$. Since Ψ is injective on abstract correspondences, it follows that the families $\{r_i\}$ and $\{r'_j\}$ consist of precisely the same members, in other words $\bigcup r_i = \bigcup r'_j$. This shows Ψ is injective as well as surjective, which, by definition, means it is a bijection, hence done. In (viii), surjectivity and symmetry-preservation at handles are both immediate. The uniqueness and everywhere-definedness are easy using (1) (for "only if" direction) and (2) (for "if" direction). ∎

2.22 Conclusion of Proof of Theorem 2.12. By Proposition 2.21 (v) and (vi), Ψ is surely a functor from PS_{rel} to PS^g_{rel}. This functor is full and faithful by (viii), and preserves all boolean operations by (i)–(iv). Thus the conditions (i) and (iii) of Proposition 2.17 are satisfied. But by Proposition 2.19, we already know that for any object $\langle \mathbf{P},\ \mathcal{G}\rangle$ in PS^g_{rel}, we can take $[\![\langle \mathbf{P},\ \mathcal{G}\rangle]\!]$ which is an object in PS_{rel}, then $\langle \mathbf{P},\ \mathcal{G}\rangle \simeq \Psi([\![\langle \mathbf{P},\ \mathcal{G}\rangle]\!])$, which gives the condition (ii) of Proposition 2.17, so that we have established the allegorical equivalence between PS_{rel} and PS^g_{rel}. But fullness and faithfulness of Ψ at the level of allegories also imply that the induced categorical functors are also full and faithful, since the fullness follows from the "if" direction of (vii) of Proposition 2.21 above, while faithfulness is immediate since any p-map is a p-relation. Thus Ψ restricted to PS gives a full and faithful categorical functor from PS to PS^g, and, together with Proposition 2.19 again (note isomorphisms coincide between PS_{rel} and PS^g_{rel}), we know PS and PS^g are categorically equivalent. ■

This proves Theorem 2.12 and establishes the formal equivalence between two presentations.

References

[1] Honda, K. Notes on P-Algebra (1): Process Strcuture. *Proc.TPPP'94*, LNCS 907, Spinger-Verlag, 1995.

[2] Mac Lane, S. *Categories for Working Mathematician.* Springer-Verlag, 1971.

[3] Milner, R. Processes: a Mathematical Model of Computing Agents. *Proc. Logic Colloquium '73.* North Holland, 1973.