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Notes on P-Algebra (3): Cartesian Product of Process Structures

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*Refereed by: Cliff Jones

Abstract

We establish a cartesian presentation of a product of an indexed family of process structures, giving an analogue of a similar presentation of set-theoretic product. Such a presentation is non-trivial due to a specific nature of the process relation in process structures and forms a basis of the algebraic theory of processes to be developed elsewhere.

1 Introduction

Let $\mathbf{P}_1, \ldots, \mathbf{P}_n$ be *process structures*, i.e. triples of a form $\langle \mathbf{P},\ \mathcal{H},\ \mathcal{S}\rangle$ where $\mathbf{P}$ is a set of *processes*, $\mathcal{H}$ is a function which assigns a finite set of *handles* to each $p \in \mathbf{P}$, and $\mathcal{S}$ gives a permutation group[1]. Then we can form an n-ary product

$$(..(\mathbf{P}_1 \times \mathbf{P}_2) \times \ldots) \times \mathbf{P}_{n-1}) \times \mathbf{P}_n \tag{1}$$

starting from the binary product given as:

$$\mathbf{P} \times \mathbf{Q} = \{\langle p,\ \delta,\ q\rangle \mid p \in \mathbf{P} \wedge q \in \mathbf{Q}\} \tag{2}$$

where δ denotes a partial one-one function from $\mathcal{H}(p)$ to $\mathcal{H}(q)$. A triple $\langle p,\ \delta,\ q\rangle$ is called a *correspondence from p to q*, where p is its *source*, δ is its *function part*, and q is its *target*. Intuitively, a process stands for an entity having multiple points of interface with the outside (like communication ports) with symmetries on them, and a correspondences is a specification of how we *connect* one process to another via their handles. The idea of symmetries suggests that two correspondences mediated by symmetries denote essentially the same situation, thus we set: $\langle p,\ \delta,\ q\rangle \sim \langle p,\ \rho_2\circ\delta\circ\rho_1,\ q\rangle$ with $\rho_1 \in \mathcal{S}(p)$ and $\rho_2 \in \mathcal{S}(q)$. Then $\sim$ is immediately an equivalence relation.

Returning to (1), in order for the definition to be meaningful, a binary product (2) itself should be a process structure, and this we achieve simply by taking the equivalence classes of $\mathbf{P} \times \mathbf{Q}$ by $\sim$ and regarding them as processes. Actually we take one representative element from each equivalence class, so that we deal with concrete correspondences instead of their equivalence classes.[2]

$$\begin{array}{rcl} \mathcal{H}(\langle p,\ \delta,\ q\rangle) & \stackrel{\text{def}}{=} & \mathcal{H}(p)\backslash\mathtt{im}(\delta^{-1}) \uplus \delta \uplus \mathcal{H}(q)\backslash\mathtt{im}(\delta) \qquad (\mathtt{im}(R) \text{ is the image of } R) \\ \mathcal{S}(\langle p,\ \delta,\ q\rangle) & \stackrel{\text{def}}{=} & \{\langle \rho_1,\ \rho_2\rangle \mid \rho_1 \in \mathcal{S}(p),\ \rho_2 \in \mathcal{S}(q),\ \text{and } \rho_2 \circ \delta \circ \rho_1 = \delta\} \end{array}$$

where $\langle \rho_1,\ \rho_2\rangle$ acts on $\mathcal{H}(\langle p,\ \delta,\ q\rangle)$ as:

$$\begin{array}{rcl} \langle \rho_1,\ \rho_2\rangle(h_1) & = & \rho_1(h_1) \quad (h_1 \in \mathcal{H}(p)\backslash\mathtt{pre}(\delta)) \\ \langle \rho_1,\ \rho_2\rangle(h_1 \mapsto h_2) & = & \rho_1(h_1) \mapsto \rho_2(h_2). \\ \langle \rho_1,\ \rho_2\rangle(h_2) & = & \rho_2(h_2) \quad (h_2 \in \mathcal{H}(q)\backslash\mathtt{im}(\delta)). \end{array}$$

[1] A permutation group is a set of bijections over a given set which contains the identity and is closed under function composition and inverse. Any group action on a set induces a permutation group of the set. Note also the empty function itself forms a (one-element) permutation group on the empty set.

[2] Thus we use a choice function. Different choice functions only result in isomorphic process structures [1]. An alternative formulation which does not rely on choice functions is given in [2].

It is easy to observe (cf. [1]) that $\mathcal{S}$ forms a permutation group over $\mathcal{H}(\langle p,\ \delta,\ q\rangle)$. Thus $\mathbf{P}\times\mathbf{Q}$ is a process structure so that (1) is well-defined. Moreover we can prove (for example as a simple corollary of Proposition 3.11 of [1]) that an n-ary product enjoys such an isomorphism law as

$$(3) \qquad (\mathbf{P}\times\mathbf{Q})\times\mathbf{R} \ \simeq\ \mathbf{P}\times(\mathbf{Q}\times\mathbf{R})$$

as well as other familiar ones. Here $\mathbf{P}\simeq\mathbf{Q}$ means $\mathbf{P}$ and $\mathbf{Q}$ are *isomorphic*, which means that there is a bijection, say F, from processes of $\mathbf{P}$ to processes of $\mathbf{Q}$, such that if $q = F(p)$ then there is a bijection θ from $\mathcal{H}(p)$ to $\mathcal{H}(q)$ with $\mathcal{S}(q) = \theta\circ\mathcal{S}(p)\circ\theta^{-1}$. Now (3) means that (1) can be written without caring the way of putting parenthesis in product expressions, so we might be tempted to write, with $I \stackrel{\text{def}}{=} 1,\ldots,n$ (or with actually any finite index set), an n-ary product as $\prod_{i\in I}\mathbf{P}_i$, just like in the case of set-theoretic products. But if we start thinking about a concrete way of defining this object, we soon realise that the idea is far from obvious, and that the nature of n-ary product in process structure is rather subtle. This is best explained by making a close look at (3), a seemingly innocent equation.

Assume given

$$(4) \qquad \langle\langle p,\ \delta_1,\ q\rangle,\ \delta_2,\ r\rangle.$$

In the light of the isomorphism law, there should exist a correspondence of the form

$$(5) \qquad \langle p,\ \delta_1',\ \langle q,\ \delta_2',\ r\rangle\rangle$$

which is expected to denote essentially the same situation as (4) denotes. Such a translation is trivial in the context of set-theoretic product, where we just change $((x,y),z)$ into $(x,(y,z))$. However an appropriate way of translating (4) to (5) may not be apparent at first sight. This is firstly because translation involves not only ordered tuples of processes but also the transformation from δ_1 and δ_2 to δ_1' and δ_2'. Moreover, two of these correspondences, δ_2 and δ_1', specify (what one may call) *higher-order connections*, in the sense that δ_2 of (4) is a connection *from* another connection, while δ_1' of (5) is a connection *to* another connection. This higher-order nature makes the conceptual grasp of such correspondences as (4) and (5) difficult, since the basic intuition that a correspondence specifies how one process is connected to another by their handles does not seem to work well. Given the difficulty, it becomes questionable what kind of objects may reside in the expression $\prod_{i\in I}\mathbf{P}_i$. Indeed, the following definition of the set-theoretic cartesian product of arbitrary indexed families does not apply to the product of processes.

> Given an I-indexed family of sets $\{X_i\}_{i\in I}$, its *cartesian product* $\prod_{i\in I} X_i$ is given by the set of all maps having a form $f : I \to \cup_{i\in I}X_i$ such that $f(i)\in X_i$ for all $i\in I$;

The definition fails to be meaningful in the present context even at the level of binary product. Thus the way of presenting $\prod_{i\in I}\mathbf{P}_i$ as a concrete object seems obscure.

The present notes however show the following.

1.1 Theorem (Main Theorem, Cartesian Product) *Given an I-indexed family of process structures $\{\mathbf{P}_i\}_{i\in I}$ with I finite, the* cartesian product of $\{\mathbf{P}_i\}_{i\in I}$, *written* $\prod_{i\in I}\mathbf{P}_i$, *is given by the set of all functions of a form* $f : (i,j) \mapsto \langle p_i,\ \delta,\ p_j\rangle \in \mathbf{P}_i \times \mathbf{P}_j$ *such that, for any i, j, k, we have:*

(i) $f(i,i) = \langle p,\ \mathbf{id}_{\mathcal{H}(p)},\ p\rangle$ for some p always ($\mathbf{id}_{\mathcal{H}(p)}$ is the identity on $\mathcal{H}(p)$).

(ii) $f(j,i) = f(i,j)^{-1}$ where we define $\langle p,\ \delta,\ q\rangle^{-1} \stackrel{\text{def}}{=} \langle q,\ \delta^{-1},\ p\rangle$.

(iii) If $f(i,j) = \langle p_i,\ \delta_{ij},\ p_j\rangle$ and $f(j,k) = \langle p'_j,\ \delta_{jk},\ p_k\rangle$, then $p_j = p'_j$. If in addition we have $f(i,k) = \langle p_i,\ \delta_{ik},\ p_k\rangle$ then δ_{ik} should satisfy the following:

$$\delta_{ik} \supset \delta_{jk} \circ \delta_{ij}. \tag{6}$$

Write $f(i)$ for the source/target of $f(i,i)$, and δ_{ij} for the relation part of $f(i,j)$. Set $f \sim f'$ iff (a) *$f(i) = f'(i)$ for each i and* (b) *$\delta'_{ij} = \rho_j^{-1} \circ \delta_{ij} \circ \rho_i$ for each (i,j) for some $\{\rho_i \in \mathcal{S}(f(i))\}_{i\in I}$. Then $\sim$ is an equivalence relation on $\prod_{i\in I}\mathbf{P}_i$ and the following data define a process structure:*

(iv) Processes: $\prod_{i\in I}\mathbf{P}_i/\sim$. We choose an arbitrary single element from each equivalence class, and use the element for its equivalence class.

(v) Handles: $\mathcal{H}(f) \stackrel{\text{def}}{=} \biguplus_{i\in I}\mathcal{H}(p_i)/\sim_f$, where $h_i \sim_f h_j \stackrel{\text{def}}{\Leftrightarrow} \delta_{ij} : h_i \mapsto h_j$.

(vi) Symmetries: $\mathcal{S}(f) \stackrel{\text{def}}{=} \{\{\rho_i\}_{i\in I} \mid \rho_i \in \mathcal{S}(p_i), \forall i,j \in I.\ \rho_j^{-1} \circ \delta_{ij} \circ \rho_i = \delta_{ij}\}$ where $\{\rho_i\}_{i\in I}$ acts on $\mathcal{H}(f)$ elementwise. Such $\{\rho_i\}_{i\in I}$ is said to be *pairwise-consistent under f.*

Moreover this process structure is isomorphic to $(..(\mathbf{P}_{i_1}\times\mathbf{P}_{i_2})\times\ldots)\times\mathbf{P}_{i_n}$, where $\{i_1, .., i_n\} = I$ with all i_k distinct, regardless of the choice of elements in (iv).

Consequently the stated isomorphism holds for any variant of the product equated by (3) and other familiar isomorphism laws. A cartesian product is also called a *pairwise decomposition of product*, for an obvious reason. Note that by $\delta_{ji} = \delta_{ij}^{-1}$ and by δ_{ii} being vacuous, we only need $\frac{n(n-1)}{2}$ decomposed correspondences to present an n-ary higher-order one. In particular, if n=3 then $\frac{n(n-1)}{2} = 3$ so that a triple of pairs of "first-order" correspondences are able to represent a higher-order correspondence of order 3.

Some observations follow.

(1) An element of a cartesian product not only gives a neutral presentation of variants of a higher-order correspondence like (4) and (5), but also gives a clear conceptual grasp about what concrete situations they describe. By the condition (6), we know that an element of a cartesian product is, essentially speaking, ordered n processes together with mutual connections (on handles) which conform to a natural consistency condition: if (say) p_1 and p_2 are connected by h_1 and h_2, and p_2 and p_3 are connected at h_2 and h_3, then p_1 and p_3 *should* be connected at h_1 and h_3.

(2) Given the geometric understanding as above, clauses (iv) (v) and (vi) of Theorem 1.1 can be understood with ease. In (iv), we like to identify configurations (processes and their connection) describing essentially the same geometric situation. In (v), those handles in the configuration which are connected to each other are identified as a single handle. Finally, in (vi), we search for a family of symmetries one from each process which, as a whole, is a symmetry of the configuration. Thus the underlying geometric picture is quite simple.

(3) As in the case of set-theoretic cartesian products, the definition easily generalises to the case where I is infinite. This becomes important when thinking about e.g. infinitary operations.

We also note that the cartesian representation constitutes an essential basis of theory of algebra over process structures. Just as in an algebra over a set, an operation in an algebra over a given process structure is a map from the n-ary product of the process structure to the process structure. Due to subtleties of n-ary product we discussed before, the theory is hard to develop without using the cartesian presentation of product in Theorem 1.1. Such applications will be developed elsewhere.

In the rest of the notes, the theorem is proved in Section 2, after necessary prerequisites. Section 3 gives a brief illustration of how to decompose (compose) a higher-order correspondence into (from) the equivalent cartesian form in a concrete way, shedding another light on the main result of the present notes.

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2 Preliminaries and Proof of Theorem 1.1

In this chapter we shall prove Theorem 1.1 after preliminaries. The main result is that $\prod_{i\in I}\mathbf{P}_i$ is a categorical product in an ambient category. Since process isomorphism coincides with isomorphism in the category, and products constructed from a binary product are readily shown to be categorical products, this proves the theorem.

We start from recollection of some notions given in [1].

2.1 Definition.

(i) A p-map $\Re$ from $\mathbf{P}$ to $\mathbf{Q}$ is a set of correspondences satisfying the following conditions:

(A). $\forall p\in\mathbf{P}.\ \exists\delta, q\in\mathbf{Q}.\ \langle p,\ \delta,\ q\rangle\in\Re$ (totality)

(B). $\langle p,\ \delta,\ q\rangle\in\Re \quad\Rightarrow\quad (\langle p,\ \delta',\ q'\rangle\in\Re \ \Leftrightarrow\ \delta\sim\delta'\wedge q=q')$ (uniqueness)

(C). $\langle p,\ \delta,\ q\rangle\in\Re \quad\Rightarrow\quad \delta$ is surjective and $\forall\rho\in\mathcal{S}(p).\exists\rho'\in\mathcal{S}(q).\delta\circ\rho=\rho'\circ\delta$.
(surjectivity and symmetry preservation of its components)

(ii) Given a p-map $\Re$, its inverse $\Re^{-1}$ is given by: $\langle q,\ \delta^{-1},\ p\rangle \in \Re^{-1}$ iff $\langle p,\ \delta,\ q\rangle \in \Re$. An *isomorphism* is a p-map whose inverse is again a p-map. The *identity* on $\mathbf{P}$, written $\mathbf{ID}_{\mathrm{P}}$, is a p-map defined by $\{\langle p,\ \rho,\ p\rangle \mid \rho \in \mathcal{S}(p), p \in \mathbf{P}\}$, which is clearly an isomorphism.

Note the symmetry preservation of $\langle p,\ \delta,\ q\rangle$ with δ surjective holds if and only if $\delta(\mathcal{S}(p))$ is a subgroup of $\mathcal{S}(q)$ (understanding $\delta(\mathcal{S}(p))$ as a group on $\mathcal{H}(q)$ induced by δ), i.e. when all the original symmetries, restricted to those handles where δ is defined, are preserved by the correspondence. Hence the name.

The following shows that having an isomorphism between two processes is equivalent to their being isomorphic in the sense given in Introduction.

2.2 Proposition. *Given two process structures* $\mathbf{P}$ *and* $\mathbf{Q}$*, there is an isomorphism between* $\mathbf{P}$ *and* $\mathbf{Q}$ *if and only if two are isomorphic, i.e. if and only if there exists a bijection* F *between processes so that there is a bijection* θ *over handles from* p *to* $F(p)$ *such that* $\mathcal{S}(F(p)) = \theta \circ \mathcal{S}(p) \circ \theta^{-1}$.

PROOF: If $\mathbf{P}$ and $\mathbf{Q}$ are isomorphic, close the bijection by symmetries on both sides, then the resulting relation certainly satisfies (A)(B)(C) of Definition 2.1 (i) with symmetry preservation coming from group isomorphism. This is a symmetric argument, hence it is a p-map in both ways, as required. Conversely, if a p-map whose inverse is a p-map exists, then clearly the function part of each correspondence is bijection. Let $p \in \mathbf{P}$ be related to q and the bijection be θ. Now we define:

$$S \stackrel{\text{def}}{=} \{\theta \circ \rho \circ \theta^{-1} \mid \rho \in \mathcal{S}(p)\}$$

and show $S = \mathcal{S}(q)$. First we show they are equal as sets. Let $s \in S$. Then $s \circ \theta = \theta \circ \rho$ since θ is a bijection. By symmetry preservation $s \circ \theta = \rho_q \circ \theta$ for some $\rho_q \in \mathcal{S}(q)$ but the equation shows $s = \rho_q$, showing $S \subset \mathcal{S}(q)$. Next take any $\rho_q \in \mathcal{S}(q)$. Let $\rho \stackrel{\text{def}}{=} \theta^{-1} \circ \rho_q \circ \theta$. Then $\rho \circ \theta^{-1} = \theta^{-1} \circ \rho_q$ and θ^{-1} is symmetry preserving, so that, just as before, we get $\rho \in \mathcal{S}(p)$, showing $S \supset \mathcal{S}(q)$. Finally two equations:

$$\begin{aligned}(\theta \circ \rho^{-1} \circ \theta^{-1}) \circ (\theta \circ \rho \circ \theta^{-1}) &= \theta \circ \theta^{-1} \\ (\theta \circ \rho \circ \rho' \circ \theta^{-1} &= (\theta \circ \rho \circ \theta^{-1}) \circ (\theta \circ \rho' \circ \theta^{-1})\end{aligned}$$

show that $\rho \mapsto \theta \circ \rho \circ \theta^{-1}$ is a homomorphism (hence an isomorphism), concluding the proof. ■

2.3 Proposition.

(i) *Given* $\Re_1 : \mathbf{P} \to \mathbf{Q}$ *and* $\Re_2 : \mathbf{Q} \to \mathbf{R}$*, its composition* $\Re_2 \circ \Re_1$ *is given by* $\{\langle p,\ \delta' \circ \delta,\ r\rangle \mid \langle p,\ \delta,\ q\rangle \in \Re_1 \wedge \langle q,\ \delta',\ r\rangle \in \Re_2\}$*. Then* $\Re_2 \circ \Re_1$ *is again a p-map.*

(ii) *Let* $\Re$ *be a p-map from* $\mathbf{P}$ *to* $\mathbf{Q}$*. Then always*

$$\Re \circ \mathbf{ID}_{\mathrm{P}} = \mathbf{ID}_{\mathrm{Q}} \circ \Re = \Re$$

(iii) *A p-map* $\Re : \mathbf{P} \to \mathbf{Q}$ *is an isomorphism if and only if the following holds:*

$$\Re \circ \Re^{-1} = \mathbf{ID}_{\mathrm{P}} \quad \text{and} \quad \Re^{-1} \circ \Re = \mathbf{ID}_{\mathrm{Q}}$$

PROOF: (i) and (ii) are from [1]. For "if" direction of (iii), we should show $\Re^{-1}$ is a p-map. Clearly it is everywhere defined and unique for processes, and every correspondence in it is surjective. Moreover since $\Re$ is closed under $\sim$ $\Re^{-1}$ is also closed under $\sim$. Thus the only remaing points is whether each correspondence is symmetry-preserving. But this is easy: let $\langle p,\ \delta,\ q\rangle \in \Re^{-1}$ and $\rho \in \mathcal{S}(p)$. By assumption we know $\delta \circ \rho \circ \delta^{-1} \in \mathbf{ID_Q}$, i.e. $\delta \circ \rho = \rho' \circ \delta$ with $\rho' \in \mathcal{S}(q)$, hence done. For "only if" direction, suppose $\langle p,\ \delta,\ q\rangle \in \Re$ and $\langle q,\ \delta',\ p\rangle \in \Re^{-1}$. Then we know $\delta' \sim \delta^{-1}$, hence $\delta' = \rho \circ \delta^{-1} \circ \rho'$ with $\rho \in \mathcal{S}(p)$ and $\rho' \in \mathcal{S}(q)$. By symmetry preservation in $\Re^{-1}$, this can be written as $\delta' = \rho'' \circ \delta^{-1}$ with $\rho'' \in \mathcal{S}(p)$ hence $\delta' \circ \delta \in \mathcal{S}(p)$. This shows that any element of $\Re^{-1} \circ \Re$ is a symmetry as far as it goes. That it covers all symmetries can be shown by composing δ with $\rho \circ \delta^{-1}$ for each $\rho \in \mathcal{S}(p)$, hence done. ■

Thus the collection of all process structures and p-maps form a category. We call the category PS. The identity of an object $\mathbf{P}$ in PS is $\mathbf{ID}_{\mathrm{P}}$, and isomorphisms in PS are precisely isomorphisms as p-maps. The following is a corollary of Proposition 3.11 of [1].

2.4 Proposition. $\mathbf{P} \times \mathbf{Q}$*, defined as in* (1.2) *and regarded as a process structure as in* (1.3) *is a categorical product of* $\mathbf{P}$ *and* $\mathbf{Q}$ *in* PS*. Since a process structure with a single process which has no handles, written* $\mathbf{1}$*, gives the obvious terminal object, we conclude that* PS *is closed under finite products.*[3]

As a consequence, as well as the isomorphisms such as $\mathbf{P} \times \mathbf{1} \simeq \mathbf{P}$ and $\mathbf{P} \times \mathbf{Q} \simeq \mathbf{Q} \times \mathbf{P}$ and the one in (3), we have the following.

2.5 Corollary. $\tilde{\mathbf{P}} \stackrel{\text{def}}{=} (..(\mathbf{P}_1 \times \mathbf{P}_2) \times \ldots) \times \mathbf{P}_{n-1}) \times \mathbf{P}_n$*, as an object of* PS*, is a categorical product of* $\{\mathbf{P}_i\}_{i \in I}$*,* $I = \{1, \ldots, n\}$*, i.e. it has a family of projections* $\{\Pi_i\}_{i \in I}$ *with* $\Pi_i : \tilde{\mathbf{P}} \to \mathbf{P}_i$*, so that if there is a family of p-maps* $\{\Re_i\}_{i \in I}$ *with* $\Re_i : \mathbf{Q} \to \mathbf{P}_i$ *for each* i*, there exists a unique p-map* Ψ *from* $\mathbf{Q}$ *to* $\mathbf{R}$ *for which the following diagram commutes for every* i*:*

$$\begin{tikzcd}
& Q \arrow[dl, "\Re_i"'] \arrow[d, "\Psi"] \\
\mathbf{P}_i & \tilde{\mathbf{P}} \arrow[l, "\Pi_i"]
\end{tikzcd}$$

[3]Actually PS is even a topos, as will be discussed elsewhere.

Our aim is to show that $\prod_{i\in I}\mathbf{P}_i$ defined as in Theorem 1.1 satisfies exactly the same property. We begin by understanding how each element of $\prod_{i\in I}\mathbf{P}_i/\sim$ can be seen as a process. We remember, from 1.1, that $\{\rho_i\}_{i\in I}$ ($\rho_i \in \mathcal{S}(p_i)$) is *pairwise-consistent under* f iff:

$$\forall i,j \in I.\rho_j^{-1}\circ\delta_{ij}\circ\rho_i = \delta_{ij}. \tag{7}$$

holds.

2.6 Proposition. *Given* $\prod_{i\in I}\mathbf{P}_i$*, fix its element* $f : (i,j) \mapsto \langle p_i,\ \delta_{ij},\ p_j\rangle \in \mathbf{P}_i \times \mathbf{P}_j$*. We let* h_i, h'_i *denote elements of the* i*-th summand of* $\uplus_{i\in I}\mathcal{H}(p_i)$ *and* $\rho_i, \rho'_i, \ldots$ *elements of* $\mathcal{S}(p_i)$*. Then we have:*

(i) $\sim_f$ *is an equivalence relation over* $\uplus_{i\in I}\mathcal{H}(p_i)$.

(ii) $h_i \sim_f h'_i$ *implies* $h_i = h'_i$.

(iii) *If* $\{\rho_i\}_{i\in I}$ *is pairwise-consistent under* f*, we have:*

$$h_i \sim_f h'_j \ \Leftrightarrow\ \rho_i(h_i) \sim_f \rho_j(h_j) \tag{8}$$

for any i *and* j.

(iv) $\langle\!\langle \{\rho_i\}_{i\in I}\rangle\!\rangle$ *is a bijection over* $\uplus_{i\in I}\mathcal{H}(p_i)/\sim$ *which is distinct if* $\{\rho_i\}_{i\in I} \neq \{\rho'_i\}_{i\in I}$.

(v) *If* $\{\rho_i\}_{i\in I}$ *and* $\{\rho'_i\}_{i\in I}$ *are pairwise consistent under* f*, then* $\{\rho'_i \circ \rho_i\}_{i\in I}$ *is again pairwise-consistent under* f *and* $\langle\!\langle \{\rho'_i\circ\rho_i\}_{i\in I}\rangle\!\rangle = \langle\!\langle \{\rho_i\}_{i\in I}\rangle\!\rangle \circ \langle\!\langle \{\rho_i\}_{i\in I}\rangle\!\rangle$*. Similarly if* $\{\rho_i\}_{i\in I}$ *is pairwise consistent under* f*, then so is* $\{\rho_i^{-1}\}_{i\in I}$ *and* $\langle\!\langle \{\rho_i^{-1}\}_{i\in I}\rangle\!\rangle = \langle\!\langle \{\rho_i\}_{i\in I}\rangle\!\rangle^{-1}$*. Finally* $\{\mathbf{id}_{p_i}\}_{i\in I}$ *is pairwise consistent under* f *and* $\langle\!\langle \{\mathbf{id}_{p_i}\}_{i\in I}\rangle\!\rangle$ *is identity on* $\uplus_{i\in I}\mathcal{H}(p_i)$.

PROOF:

(i) That $\sim_f$ is equivalence is a direct consequence of (i)(ii)(iii) of Theorem 1.1. In detail: $h_i \sim_f h_i$ because $f(i,i) = \langle p_i,\ \mathbf{id}_{p_i},\ p_i\rangle$. $h_i \sim_f h'_j$ implies $h'_j \sim_f h_i$ because $f(i,j) = f(j,i)^{-1}$. Finally if $h_i \sim_f h'_j$ and $h'_j \sim_f h''_k$ then $\delta_{ij} : h_i \mapsto h'_j$ and $\delta_{jk} : h'_j \mapsto h''_k$ but $\delta_{ik} \supset \delta_{jk}\circ\delta_{ij}$ so that $\delta_{ik} : h_i \mapsto h''_k$, i.e. $h_i \sim h''_k$, as required.

(ii) Suppose, contrarily, we have $h_i \sim_f h'_i$ but $h_i \neq h'_i$. Then $h_i \mapsto h'_i$ for δ_{ii} but this contradicts $\delta_{ii} = \mathbf{id}_p$, hence as required.

(iii) ($\Rightarrow$) If $i = j$, the result is obvious from the above (ii). If $i \neq j$, then by definition $\delta_{ij}\circ\rho_i(h_i) = \rho_j\circ\delta_{ij}(h_i)$ hence: $\delta_{ij} : \rho_i(h_i) \mapsto \rho_j(h'_j)$, I.e. $\rho(h_i) \sim_f \rho(h'_j)$, as required.
($\Leftarrow$) Just trace the reasoning above backward.

(iv) That $\langle\!\langle \{\rho_i\}_{i\in I}\rangle\!\rangle$ is a bijection is immediate from (iii) above. For distinctness we use (ii) above. If $\{\rho_i\}_{i\in I} \neq \{\rho'{}_i\}_{i\in I}$ then for some i and h_i we have $\rho_i(h_i) \neq \rho'_i(h_i)$. If, in spite of this, we have $\langle\!\langle \{\rho_i\}_{i\in I}\rangle\!\rangle([h_i]_\sim) = \langle\!\langle \{\rho'{}_i\}_{i\in I}\rangle\!\rangle([h_i]_\sim)$, then for some $h'_i \neq h''_i$ we have $h'_i \sim_f h''_i$, contradicting (ii).

(v) Suppose $\{\rho_i\}_{i\in I}$ and $\{\rho'_i\}_{i\in I}$ are pairwise consistent under f. Then (7) holds for ρ_i, ρ_j and ρ'_i, ρ'_j, so that:

$$\begin{aligned}(\rho_j \circ \rho'_j)^{-1} \circ \delta_{ij} \circ (\rho_i \circ \rho'_i) &= {\rho'_j}^{-1} \circ {\rho_j}^{-1} \circ \delta_{ij} \circ \rho_i \circ \rho'_i \\ &= {\rho'_j}^{-1} \circ \delta_{ij} \circ \rho'_i \\ &= \delta_{ij}\end{aligned}$$

as required. Moreover, by (iv), $\langle\!\langle \{\rho_i\}_{i\in I} \rangle\!\rangle$ is a function so that we can write: $\langle\!\langle \{\rho'_i \circ \rho_i\}_{i\in I} \rangle\!\rangle([h_j]_\sim) = \langle\!\langle \{\rho'_i\}_{i\in I} \rangle\!\rangle([\rho_j(h_j)]_\sim) = \langle\!\langle \{\rho'_i\}_{i\in I} \rangle\!\rangle(\langle\!\langle \{\rho_i\}_{i\in I} \rangle\!\rangle([h_j]_\sim))$ hence done. For the inverse,

$$\begin{aligned}\rho_j^{-1} \circ \delta_{ij} \circ \rho_j = \delta ij \quad &\Leftrightarrow \quad (\rho_j^{-1} \circ \delta_{ij} \circ \rho_i)^{-1} = \delta_{ij}^{-1} \\ &\Leftrightarrow \quad \rho_i^{-1} \circ \delta_{ji} \circ \rho_j^{-1} = \delta_{ji},\end{aligned}$$

and $\langle\!\langle \{\rho_i^{-1}\}_{i\in I} \rangle\!\rangle([h_j]_\sim) = [\rho_j^{-1}(h_j)]_\sim = \langle\!\langle \{\rho_i\}_{i\in I} \rangle\!\rangle^{-1}([h_j]_\sim)$. Finally $\{\mathbf{id}_{p_i}\}_{i\in I}$ is pairwise consistent under any f and it clearly induces identity on $\uplus_{i\in I}\mathcal{H}(p_i)/\sim_f$, hence done. ■

By (v) of Proposition 2.6, $\{\langle\!\langle \{\rho_i\}_{i\in I} \rangle\!\rangle \mid \{\rho_i\}_{i\in I}$ is pairwise consistent under $f\}$ forms a permutation group on $\uplus_{i\in I}\mathcal{H}(p_i)/\sim_f$. Thus (iv)(v)(vi) of Theorem 1.1 do define a process structure. We also check:

2.7 Proposition. *In the process structure given by* (iv)(v)(vi) *of Theorem 1.1, different elements from the same equivalence class in* $\prod_{i\in I}\mathbf{P}_i/\sim$ *result in isomorphic process structures.*

Proof: Suppose $f \sim f'$. We first form a bijection from $\mathcal{H}(f)$ to $\mathcal{H}(f')$. Suppose δ_{ij} and δ'_{ij} are components of $f(i,j)$ and $f'(i,j)$, respectively. Then we can set $\delta_{ij} \circ \theta_i = \theta_j \circ \delta'_{ij}$ with $\theta_i \in \mathcal{S}(p_i)$ and $\theta_j \in \mathcal{S}(p_j)$, so that $\delta'_{ij} : h_i \mapsto h_j$ if and only if $\delta_{ij} : \theta_i(h_i) \mapsto \theta_j(h_j)$, which clearly gives a bijection between $\uplus_{i\in I}\mathcal{H}(p_i)/\sim_f$ and $\uplus_{i\in I}\mathcal{H}(p_i)/\sim_{f'}$. Under the same assumption, we construct the isomorphism as follows: given $\{\rho_i\}_{i\in I} \in \mathcal{S}(f)$, let $\rho'_i \stackrel{\mathrm{def}}{=} \theta_i^{-1} \circ \rho_i \circ \theta_i$ for each I. Then $\{{\rho'}_i\}_{i\in I}$ indeed defines a pairwise consistent symmetries under f', and the mapping is bijective and homomorphic. We omit the details. ■

Having finished the first part of the proof, now we construct projection p-maps from $\prod_{i\in I}\mathbf{P}_i$ to each $\mathbf{P}_i$.

2.8 Definition and Proposition. (projection maps) *Given* $\prod_{i\in I}\mathbf{P}_i$ *regarded as a process structure by data specified in* (iv)(v)(vi) of 1.1, *the* i-th projection of $\prod_{i\in I}\mathbf{P}_i$, *often written* Π_i, *is given by the set of correspondences, whose typical element is* $\langle f,\ \pi_i,\ p_i\rangle$ *with* f *being a process from* $\prod_{i\in I}\mathbf{P}_i$ *and* p_i *from* $\mathbf{P}_i$, *such that:*

$$\pi_i : [h]_{\sim_f} \mapsto \rho_i(h_i) \quad \Leftrightarrow \quad h_i \in [h]_{\sim_f} \cap \mathcal{H}(p_i) \tag{9}$$

for some $\rho_i \in \mathcal{S}(p_i)$. Then Π_i gives a p-map for each i.

PROOF: We first note that (9) is well-defined, i.e. gives a one-one partial function, due to the fact that, by 2.6 (ii), in each $[h]_{\sim_f}$ of $\mathcal{H}(f)$, there is only one element (if any) from $\mathcal{H}(p_i)$ for each i (by writing "$[h]_{\sim_f} \cap \mathcal{H}(p_i)$" we are confusing $\mathcal{H}(p_i)$ and the i-th component of $\uplus_{i\in I}\mathcal{H}(p_i)$ but this should be hardly misleading).

We now show Π_i is a p-map. First clearly Π_i is defined for each process from $\prod_{i\in I}\mathbf{P}_i$ (as a process structure). Suppose $\langle f,\ \pi_i,\ p_i\rangle \in \Pi_i$. Then by Proposition 2.6 (ii) again, we know $\langle f,\ \pi_i',\ p_i'\rangle \in \Pi_i$ implies $p_i = p_i'$. Next, noting each π_i of (9) comes from an element of $\mathcal{S}(p_i)$, if $\langle f,\ \pi_i,\ p_i\rangle, \langle f,\ \pi_i',\ p_i\rangle \in \Pi_i$, with π_i coming from ρ_i and π_i' is from ρ_i', then $\pi_i' = \rho'^{-1} \circ \rho_1 \circ \pi_i \sim \pi_i$. Conversely if $\pi_i \sim \pi_i''$ for some π_i'' then for some $\rho' \in \mathcal{S}(p_i)$ and $\rho \in \mathcal{S}(f)$ we have $\pi_i'' = \rho' \circ \pi_i \circ \rho$ but since π_i'' is defined only on $\mathcal{H}(p_i)$ we can regard ρ as acting on $\mathcal{H}(p_i)$, i.e. as an element of $\mathcal{S}(p_i)$, thus π_i'' is again such, which implies $\langle f,\ \pi_i'',\ p_i\rangle \in \pi_i$. Finally we should show that if $\langle f,\ \pi_i,\ p_i\rangle$ then π_i is surjective and symmetry preserving. But surjectivity is immediate since $\sim_f$ partitions $\uplus_{i\in I}\mathcal{H}(p_i)$ such that no $h_i, h_i' \in \mathcal{H}(p_i)$ are in the same partition. For symmetry preservation, suppose $\rho \in \mathcal{S}(f)$. Then $\rho' \stackrel{\text{def}}{=} \rho \restriction \mathcal{H}(p_i) \in \mathcal{S}(p_i)$ as noted, hence if $\pi_i : \rho_i(h_i) \mapsto h_i$ from $\rho_i \in \mathcal{S}(p_i)$,

$$\delta \circ \rho = \delta \circ \rho' = \rho_i \circ \rho' \circ \rho_i \circ \delta.$$

But $\rho_i \circ \rho' \circ \rho_i \in \mathcal{S}(p_i)$, hence done. ■

The main result of this chapter follows.

2.9 Lemma. (universality of projection p-maps) *Given $\prod_{i\in I}\mathbf{P}_i$ and the family of projection p-maps $\{\Pi_i\}_{i\in I}$, if, for some $\mathbf{Q}$, we have a family of p-maps $\{\Re_i\}_{i\in I}$ each from $\mathbf{Q}$ to $\mathbf{P}_i$, then there is a unique Ψ such that the following diagram holds for every i:*

(10)
$$\begin{tikzcd} & \mathbf{Q} \arrow[dl, "\Re_i"'] \arrow[d, "\Psi"] \\ \mathbf{P}_i & \prod_{i\in I}\mathbf{P}_i. \arrow[l, "\Pi_i"] \end{tikzcd}$$

To prove Lemma 2.9, the construction of Ψ is crucial. We start with forming each correspondence of Ψ, by integrating information from each element of $\Re_i$.

2.10 Definition. Let $q \in \mathbf{Q}$ and $\{\Re_i\}_{i\in I}$ with $\Re_i$ from $\mathbf{Q}$ to $\mathbf{P}_i$. Given $\xi = \{\langle q,\ \delta_i,\ p_i\rangle\}_{i\in I}$ each from $\Re_i$, we say $f \in \prod_{i\in I}\mathbf{P}_i$ is *consistent with ξ up to $\{\rho_i\}_{i\in I}$* if the following holds: for every $i, j \in I$, and writing δ_{ij} for the component of $f(i,j)$:

(11) $\quad \delta_{ij} : h_i \mapsto h_j \quad \Leftrightarrow \quad \exists h \in \mathcal{H}(q) : \delta_i : h \mapsto \rho_i^{-1}(h_i) \ \wedge\ \delta_j : h \mapsto \rho_j^{-1}(h_j)$

for some $\{\rho_i\}_{i\in I}$ s.t. $\rho_i \in \mathcal{S}(p_i)$.

2.11 Proposition. *For any $\xi \stackrel{\text{def}}{=} \langle q,\ \delta_i,\ p_i\rangle_{i\in I}$ each from $\Re_i$, there exists $f \in \prod_{i\in I} \mathbf{P}_i$ which is consistent with ξ and, moreover, if f' is also consistent with ξ, $f \sim f'$. Thus regarding $\prod_{i\in I} \mathbf{P}_i$ as a process structure, we have a unique f satisfying the condition, which we write $\overline{\xi}$. Suppose $\overline{\xi}$ is consistent with ξ up to $\{\rho_i\}_{i\in I}$. Fix some $\{\rho'_i\}_{i\in I} \in \mathcal{S}(\overline{\xi})$ and define:*

$$\delta : h_q \mapsto [h']_{\sim_{\overline{\xi}}} \quad \stackrel{\text{def}}{\Leftrightarrow} \quad \exists i. \rho'_i \circ \rho_i \circ \delta_i(h_q) \in [h']_{\sim_{\overline{\xi}}}. \tag{12}$$

Then δ is a one-to-one, surjective, and symmetry preserving partial function from $\mathcal{H}(q)$ to $\mathcal{H}(\overline{\xi})$, and moreover satisfies, for some $\{\pi_i\}_{i\in I}$ with $\langle \overline{\xi},\ \pi_i,\ p_i\rangle \in \Pi_i$ for each i:

$$\delta_i = \pi_i \circ \delta.$$

In diagram:

$$\begin{tikzcd}
 & q \arrow[dl, "\delta_i"'] \arrow[d, "\delta"] \\
p_i & \overline{\xi}. \arrow[l, "\pi_i"]
\end{tikzcd} \tag{13}$$

Note that objects are processes and arrows are correspondences in the above diagram.

Proof: We first show that f which is consistent with ξ does exist and such f is unique up to $\sim$.

(i) (existence) We show that taking $\{\mathbf{id}_{p_i}\}_{i\in I} \in \mathcal{S}(\overline{\xi})$ as $\{\rho_i\}_{i\in I}$ in (11), the resulting f is what is required. First, since each δ_i is surjective, for each $h_i \in \mathcal{H}(p_i)$ there exists a unique $h' \in \mathcal{H}(q)$ such that $\delta_i : h' \mapsto h_i$. Therefore δ_{ij} is uniquely determined. Now $\delta_{jk} \circ \delta_{ij} : h_i \mapsto h_k$ implies for some h', h_j we have $\delta_i : h' \mapsto h_i$ and $\delta_j : h' \mapsto h_j$, and $\delta_k : h' \mapsto h_k$, so that we have $\delta_{ik} : h_i \mapsto h_k$, i.e.

$$\delta_{ik} \supset \delta_{jk} \circ \delta_{ij}.$$

Next $\delta_{ji} = \delta_{ij}^{-1}$ is trivial. Finally, setting $\delta_{ii} = \langle p_i,\ \mathbf{id}_{p_i},\ p_i\rangle$ for each i, we now have $f : (i, j) \mapsto \delta_{ij}$ which is an element of $\prod_{i\in I} \mathbf{P}_i$.

(ii) (uniqueness) Suppose f is consistent with ξ up to $\{\rho_i\}_{i\in I}$ and f' is consistent with ξ up to $\{\rho'_i\}_{i\in I}$, then easily (draw a diagram) one gets:

$$\begin{aligned}
\delta'_{ij} &= \rho'^{-1}_j \circ \rho_j \circ \delta_{ij} \circ \rho_i^{-1} \circ \rho'_i \\
&= (\rho_j \circ \rho'^{-1}_j)^{-1} \circ \delta_{ij} \circ \rho_i^{-1} \circ \rho'_i
\end{aligned}$$

for each i, j which shows (since $\rho_i^{-1} \circ \rho'_i \in \mathcal{S}(p_i)$) $f \sim f'$ by the definition of $\sim$ in 1.1.

Next we show δ given by (12) is one-to-one, surjective, and symmetry preserving function.

(i) (δ is a one-one partial function) First we show δ is a partial function, i.e. if the source handle is determined then the target handle is uniquely determined. This means, in the context of (12),

$$\rho'_i \circ \rho_i \circ \delta_i : h_q \mapsto h_i \wedge \rho'_j \circ \rho_j \circ \delta_j : h_q \mapsto h_j \quad \Rightarrow \quad h_i \sim_f h_j.$$

But $\overline{\xi}$ is consistent with ξ up to $\{\rho_i\}_{i\in I}$ so that $\delta_{ij} : \rho_i \circ \delta_i(h_q) \mapsto \rho_j \circ \delta_j(h_q)$ always, i.e. $\rho_i \circ \delta_i(h_q) \sim \rho_j \circ \delta_j(h_q)$. Since $\{\rho'_i\}_{i\in I}$ is pairwise consistent under f, by Proposition 2.6 (iii) we have $h_i \sim_f h_j$ again, as required.

Next we show that δ is one-one. Suppose $\delta : h_q \mapsto [h']_{\sim_{\overline{\xi}}}$ and $\delta : h'_q \mapsto [h']_{\sim_{\overline{\xi}}}$. Then we have, with $h_i \sim_{\overline{\xi}} h_j$, $\rho'_i \circ \rho_i \circ \delta_i : h \mapsto h_i$ and $\rho'_j \circ \rho_j \circ \delta_j : h'' \mapsto h_j$, i.e. $\rho_i \circ \delta_i : h_q \mapsto \rho'^{-1}_i(h_i) = h'_i$ and $\rho_j \circ \delta_j : h'_q \mapsto \rho'^{-1}_j(h_j) = h'_j$, But $h_i \sim_{\overline{\xi}} h_j$ means that $h'_i \sim_{\overline{\xi}} h'_j$ by Proposition 2.6 (ii), and, therefore, by (11), for some h''_q, we have $\rho_i \circ \delta_i(h''_q) = h'_i$ and $\rho_j \circ \delta_j(h''_q) = h'_j$. Since $\rho_i \circ \delta_i$ and $\rho_j \circ \delta_j$ are both partial function, we know $h_q = h''_q$ and $h'_q = h''_q$, i.e. $h'_q = h''_q$, as required.

(ii) (δ is surjective) Direct from the surjectivity of each δ_i.

(iii) (δ is symmetry preserving) Let δ'_i stand for $\rho_i \circ \delta_i$ for each i. Note δ'_i is symmetry preserving and surjective since δ_i is. Now suppose $\rho \in \mathcal{S}(q)$ and let $\delta \circ \rho : h \mapsto [h']_\sim$, i.e. for some $h_i \in [h']_\sim$ we have

$$\rho'_i \circ \delta'_i \circ \rho : h \mapsto h_i.$$

Since δ'_i is symmetry preserving, we have:

(14) $$\delta'_i \circ \rho = \rho''_i \circ \delta'_i$$

such that $\rho''_i \in \mathcal{S}(p_i)$. Such ρ''_i is unique since δ'_i is surjective. We now show that $\{\rho''{}_i\}_{i\in I}$, the collection of all such ρ_i, is in $\mathcal{S}(\overline{\xi})$. If it is so, since $\{\rho'_i\}_{i\in I} \in \mathcal{S}(\overline{\xi})$ by definition, we have $\{\rho'_i\}_{i\in I} \circ \{\rho''_i\}_{i\in I} \in \mathcal{S}(\overline{\xi})$. But we have

$$\delta \circ \rho = (\{\rho'_i\}_{i\in I} \circ \{\rho''_i\}_{i\in I}) \circ \delta.$$

so that δ is symmetry preserving.
To show $\{\rho''{}_i\}_{i\in I}$ is in $\mathcal{S}(\overline{\xi})$, i.e. is pairwise consistent under $\overline{\xi}$, suppose $\delta_{ij} : h_j \mapsto h_j$. Then for some $h \in \mathcal{H}(q)$, $\delta'_i : h \mapsto h_j$, $\delta'_j : h \mapsto h_j$. Take (unique) $\rho''_i \in \mathcal{S}(p_i)$ and $\rho''_j \in \mathcal{S}(p_j)$ such that $\delta'_i \circ \rho = \rho''_i \circ \delta'_i$ and $\delta'_j \circ \rho = \rho''_j \circ \delta'_j$. Let $\rho^{-1}(h) = h'$, $\rho''^{-1}_i(h_i) = h'_i$ and $\rho''^{-1}_j(h_j) = h'_j$. Then we have the following diagram:

(15)
$$\begin{tikzcd}
h_i \arrow[dd, "\delta_{ij}"'] & & h'_i \arrow[ll, "\rho''_i"'] \arrow[dd, "\delta_{ij}"] & \\
& h \arrow[ul, "\delta'_i"'] \arrow[dl, "\delta'_j"] & & h' \arrow[ll, "\rho"'] \arrow[ul, "\delta'_i"'] \arrow[dl, "\delta'_j"] \\
h_j & & h'_j \arrow[ll, "\rho''_j"] &
\end{tikzcd}$$

Since ρ_i and ρ_j are bijective, we have

$$\delta'_{ij} = \rho_j^{-1} \circ \delta'_{ij} \circ \rho_i$$

which shows that $\{\rho''_i\}_{i \in I}$ is pairwise consistent under $\overline{\xi}$, hence done.

Finally we show δ of (12) satisfies the diagram (13). Let $\delta_i : h \mapsto h_i$ and $\delta : h \mapsto [h_j]_\sim$. Then by definition $\rho(h_i) \in [h_j]_\sim$, so if we take $\pi_i : [h_j]_\sim \mapsto \rho_i(h_i)$ for $\{\rho_i\}_{i \in I}$ pairwise consistent under $\overline{\xi}$, i.e. $\pi_i \in \Pi_i$, and clearly (14) holds for each $h \in \mathcal{H}(q)$, as required. ■

We now establish Lemma 2.9.

2.12 Proof of Lemma 2.9. Take Ψ as the set of all $\langle q,\ \delta,\ \overline{\xi}\rangle$ constructed as (14), each $\overline{\xi}$ uniquely determined when we regard $\prod_{i \in I} \mathbf{P}_i$ as a process structure, by Proposition 2.11. We first show Ψ satisfies the condition of Lemma, then show no other p-maps from $\mathbf{Q}$ to $\prod_{i \in I} \mathbf{P}_i$ can do so.

To show that Ψ satisfies the diagram (10), we have only to show Ψ is a p-map [because: if so, $\Pi_i \circ \Psi$ is a p-map since Π_i is a p-map as we proved in Proposition 2.8: then, since $\Pi_i \circ \Psi \supset \Re_i$ from Proposition 2.11, and $\Re_i$ is everywhere defined, this implies $\Pi_i \circ \Psi = \Re_i$ for each i, thus the diagram (10) commutes]. Now suppose we have

$$\xi \stackrel{\text{def}}{=} \{\langle q,\ \delta_i,\ p_i\rangle\}_{i \in I}$$

and

$$\xi' \stackrel{\text{def}}{=} \{\langle q,\ \delta'_i,\ p'_i\rangle\}_{i \in I}$$

Since $\Re_i$ for each i is a p-map, immediately $p_i = p_i$ and $\delta'_i = \rho_i \circ \delta_i \circ \rho$ for $\rho \in \mathcal{S}(q)$ and, $\rho_i \in \mathcal{S}(p_i)$. But such δ'_i is symmetry preserving so that we can also note $\delta'_i = \rho'_i \circ \delta_i$ for $\rho'_i \in \mathcal{S}(p_i)$ for each i. Therefore we have $\overline{\xi} \sim \overline{\xi'}$, which implies $\overline{\xi} = \overline{\xi'}$. Moreover by the diagram (15), $\langle q,\ \delta,\ \overline{\xi}\rangle, \langle q,\ \delta',\ \overline{\xi}\rangle \in \Psi$ implies $\delta = \{\rho_i\}_{i \in I} \circ \delta$ for $\rho_i \in \mathcal{S}(p_i)$, therefore we have $\delta \sim \delta'$.

Suppose $\langle q,\ \delta,\ \overline{\xi}\rangle, \langle q,\ \delta,\ \overline{\xi'}\rangle \in \Psi$. Then clearly δ and δ' are consistent as (14) taking $\{\delta_i\}_{i \in I} \neq \{\delta'{}_i\}_{i \in I}$, each pairwise consistent with, or a symmetry of, $\overline{\xi}$. Now

$$\delta' = \{\rho'_i\}_{i \in I} \circ \{\rho_i^{-1}\}_{i \in I} \circ \delta$$

But $\{\rho'{}_i\}_{i \in I} \circ \{\rho^{-1}{}_i\}_{i \in I}$ is clearly pairwise consistent with $\overline{\xi}$, i.e. in $\mathcal{S}(\overline{\xi})$, hence $\delta \sim \delta'$. Suppose next that $\langle q,\ \delta,\ \overline{\xi}\rangle \in \Psi$ and $\delta' = \{\rho_i\}_{i \in I} \circ \delta \circ \rho$ where $\rho \in \mathcal{S}(q)$ and $\{\rho_i\}_{i \in I} \in \mathcal{S}(\overline{\xi})$. By symmetry preservation of δ, we can write:

$$\delta' = \{\rho_i\}_{i \in I} \circ \{\rho'_i\}_{i \in I} \circ \rho$$

so that $\langle q,\ \delta',\ \xi\rangle \in \Psi$ hence Ψ defines a p-map.

Next we show Ψ is the only p-map satisfying the property. But suppose a p-map Ψ' also satisfies the property. Then, for each $\{\pi_i \in \Pi_i\}_{i\in I}$ and $\{\delta_i \in \Re_i\}_{i\in I}$ Ψ' should contain δ which satisfies the following:

(16)
$$\begin{tikzcd} & q \arrow[dl, "\delta_i"'] \arrow[d, "\delta"] \\ p_i & f. \arrow[l, "\pi_i"] \end{tikzcd}$$

In particular, for simplicity, take $\rho_i = \mathbf{id}_{p_i}$ for each i in (9) to get $\{\pi_i\}_{i\in I}$. Then we easily know:

$$\delta : h \mapsto [h']_f \quad \Leftrightarrow \quad \exists i.\delta_i : h \mapsto h_i \in [h']_f.$$

and that

$$h_i \sim_f h_j \quad \Leftrightarrow \quad \exists h.\delta_i : h \mapsto h_i \wedge \delta_j : h \mapsto h_j.$$

This shows that $\delta \in \Psi$. But if we take other $\{\pi_i\}_{i\in I}$ (i.e. mediated by some symmetries) then by symmetry preservation we easily know that, again taking appropriate $\{\delta_i\}_{i\in I}$, we get exactly the same scheme as above. Moreover any element of Ψ' should satisfy (16) for some $\{\pi_i\}_{i\in I}$ and $\{\delta_i\}_{i\in I}$, hence all elements of Ψ' are in Ψ, which shows $\Psi' = \Psi$. ■

This establishes Lemma 2.9 so that $\prod_{i\in I} \mathbf{P}_i$ is a categorical product of $\{\mathbf{P}_i\}_{i\in I}$ in PS, hence, together with Corollary 2.5 and the standard result in category theory, we have now proved Theorem 1.1.

3 Concrete Procedure of Decomposition

In this chapter we illustrate concrete ways of translating higher-order correspondence to its cartesian presentation and, conversely, composing a higher-order correspondence from a given cartesian form. While the proof in Section 2 does tell us that any higher-order correspondence such as (4) and (5) in Introduction has its cartesian presentation, we have not (at least explicitly) mentioned how we can do such decomposition concretely. Moreover, since an isomorphism is both-ways, it should be possible to compose a higher-order correspondence from its cartesian presentation. Note that, given such procedures, it also becomes possible to compare two higher-order correspondences, like (4) and (5), and also to translate between these two forms. We discuss briefly how these things can be done. Indeed they are quite simple. We start from the decomposition procedure.

3.1 Decomposition. To be concrete, we take the 3-ary case, which easily extends to the general n-ary case. We write $\mathbf{P}^3$ for the cartesian product of three $\mathbf{P}$'s, and $\mathbf{P}\times(\mathbf{P}\times\mathbf{P})$ for the product whose elements are higher-order correspondences of a form e.g. (5). To be specific, let the element of $\mathbf{P}\times(\mathbf{P}\times\mathbf{P})$ be given as follows:

(17)
$$\underline{\delta'} \stackrel{\text{def}}{=} \langle p_1,\ \delta',\ \langle p_2,\ \delta_{23},\ p_3\rangle\rangle \ \in\ \mathbf{P}\times(\mathbf{P}\times\mathbf{P})$$

Here we denote the correspondence by underlining its function part. This convention will be used throughout the chapter.

The decomposition uses projection maps from $\underline{\delta'}$ at the level of correspondences (*not* at the level of p-maps), constructed from binary projection: to make the reasoning easier, we take the following canonical construction.

3.2 Definition. (canonical binary projections) Given $\underline{\delta} \stackrel{\mathrm{def}}{=} \langle p,\ \delta,\ q\rangle$, we define a correspondence $\langle \underline{\delta},\ \pi_1,\ p_1\rangle$ such that, given an equivalence $\sim_\delta$ over $\mathcal{H}(p_1) \uplus \mathcal{H}(p_2)$ generated from $h_1 \sim h_2 \ \Leftrightarrow\ \delta : h_1 \mapsto h_2$:

$$\pi_1 : [h_1]_{\sim_\delta} \mapsto h_1.$$

Similarly we define:

$$\pi_2 : [h_2]_{\sim_\delta} \mapsto h_2.$$

π_1 and π_2 are called *canonical binary projections of* $\underline{\delta}$.

Canonical projections are always in the binary projection p-maps of the underlying product. Using the notion, we get the following diagram by projecting $\underline{\delta'}$ of (17).

$$\begin{tikzcd}
 & \underline{\delta'} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi'"] & & \\
p_1 & & \underline{\delta_{23}} \arrow[dl, "\pi'_2"'] \arrow[dr, "\pi'_3"] & \\
 & p_2 & & p_3
\end{tikzcd}$$

Here $\langle \pi_1,\ \pi'\rangle$ are canonical projections from $\underline{\delta'}$ and $\langle \pi'_2,\ \pi'_3\rangle$ are those from $\underline{\delta_{23}}$. Thus we get *three* projections from $\underline{\delta'}$: π_1, which is from $\underline{\delta'}$ to p_1, $\pi'_2 \circ \pi'$, which is from $\underline{\delta'}$ to p_2, and $\pi'_3 \circ \pi'$, which is from $\underline{\delta'}$ to p_3. We write π_2 and π_3 for the last two projections. They are called *canonical* 3-*ary projections from* $\underline{\delta'}$ (note, similarly for an element of n-ary product, we have canonical n-ary projections). We can now present the decomposition procedure.

3.3 Proposition. *Given* $\underline{\delta'}$ *of* (17) *and their canonical* 3-*ary projections* π_1, π_2, *and* π_3, *the family* $\{\delta_{ij}\}_{i,j\in I}$ *defined by the following diagram,* i, j *ranging over* $1, 2, 3$, *is called the* canonical cartesian decomposition of $\underline{\delta'}$. *Then it is an element of* $\mathbf{P}^3$ *and is related to* $\underline{\delta'}$ *by an isomorphism from* $\mathbf{P} \times (\mathbf{P} \times \mathbf{P})$ *to* $\mathbf{P}^3$.

$$\begin{tikzcd}
 & \underline{\delta'} \arrow[dr, "\pi_j"] & \\
p_i \arrow[ur, "\pi_i^{-1}"] \arrow[rr, "\delta_{ij}"] & & p_j
\end{tikzcd}$$

PROOF: (outline) We first check that it is indeed an element of $\mathbf{P}^3$. Since $\delta_{ii} = \mathbf{id}_{\mathcal{H}(p_i)}$ and $\delta_{ji} = \delta_{ij}^{-1}$ are immediate, we are only to check $\delta_{ik} \supset \delta_{jk} \circ \delta_{ij}$ for each distinct i, j, k, which follows from the following reasoning:

$$\begin{aligned} \delta_{jk} \circ \delta_{ij} &= \pi_k \circ \pi_j^{-1} \circ \pi_j \circ \pi_i{}^{-1} \\ &\subset \pi_k \circ \mathbf{id}_{\mathcal{H}(p_j)} \circ \pi_i{}^{-}1 \\ &= \delta_{ik} \end{aligned}$$

Next we should check that $\{\delta_i\}_{i \in I}$ indeed copies all essential information content of $\underline{\delta'}$ and no other. This is verified by showing that these two as processes are related by symmetry-preserving bijection. We only mention how the bijection is constructed: it is given as $\bigcup_{1 \le i \le 3}(\pi_i^{-1} \circ \pi_i')$. That it satisfies the required properties is easy by diagram chasing. ■

Thus once projections are calculated, we just take their differences and we get the desired decomposition. As readily seen, other "symmetry variants" of the cartesian form can be constructed if we use symmetry variants of canonical projections, given by multiplying each δ_i by a symmetry of targets.

3.4 Composition. Next we give a brief account of how to reconstruct a higher-order correspondence like (17). Suppose given $\{\delta_{ij}\}_{i,j \in I} \in \mathbf{P}^3$. To make this into a form of (17), what we do is to *merge* δ_{12} and δ_{13} to get a higher-order connection δ' from p_1 to δ_{23}. We again use projections.

3.5 Proposition. *Given $\{\underline{\delta_{ij}}\} \in \mathbf{P}^3$ where each $\underline{\delta}_{ij}$ is from p_i to p_j, the* merge *of δ_{ij} and δ_{ik}, where i, j, k all distinct, is given by:*

$$\delta' \stackrel{\text{def}}{=} (\pi_j'^{-1} \circ \delta_{ij}) \cup (\pi_k'^{-1} \circ \delta_{ik})$$

where δ_j' and δ_k' are canonical projections from δ_{jk} to p_i and p_k. Then $\langle p_i,\ \delta',\ \underline{\delta_{jk}}\rangle$ is a correspondence. Moreover if we take its canonical cartesian decomposition we get the original $\{\underline{\delta_{ij}}\}$.

Remark. The last clause says that the original cartesian form and the resulting correspondence is related by an isomorphism, because, by Proposition 3.3, we know a higher-order correspondence and its cartesian form is related by an isomorphism, and the inverse of an isomorphism is an isomorphism.

PROOF: (outline) That δ' does define a higher-order connection from p_i to $\underline{\delta_{jk}}$, i.e. a partial injection from $\mathcal{H}(p_i)$ to $\mathcal{H}(\underline{\delta_{jk}})$, is verified by the following diagram:

$$\begin{tikzcd}
 & p_i \arrow[d, phantom, "\supset"] & \\
p_j \arrow[ur, "\delta_{ji}"] \arrow[rr, "\delta_{jk}"] \arrow[dr, "\pi_j'^{-1}"'] & & p_k \arrow[ul, "\delta_{ki}"'] \\
 & \underline{\delta_{jk}} \arrow[ur, "\pi_k'"'] &
\end{tikzcd}$$

where $\supset$ denotes that the commutativity is up to set inclusion. By the diagram, we know

$$\delta_{ki} \circ \pi_k' \circ \pi_j'^{-1} \subset \delta_{ji}. \tag{18}$$

But if for some $h \in \mathcal{H}(\underline{\delta_{jk}})$ we have $\pi_j'^{-1} \circ \delta_{ij} : h' \mapsto h$ and $\pi_k'^{-1} \circ \delta_{ik} : h'' \mapsto h$ and $h' \neq h''$, then (18) cannot happen. Moreover if for some $h \in \mathcal{H}(p_i)$, we have $\pi_j'^{-1} \circ \delta_{ij} : h \mapsto h'$ and $\pi_k'^{-1} \circ \delta_{ik} : h \mapsto h'' \neq h'$, then we have $\delta_{ki} \circ \pi_k' : h' \not\mapsto h$ I.e. h' is not mapped at all or mapped to a handle other than h, which again contradicts (18).

To show that the decomposition of the resulting higher-order correspondence does give the original cartesian form is immediate by looking at the situation by diagrams: the following verifies this in the case of δ_{ij}.

$$\begin{tikzcd}
 & \underline{\delta'} \arrow[dr, "\pi'"] & & \\
p_i \arrow[ur, "\pi_i'^{-i}"] \arrow[rr, "\delta'"] & & \underline{\delta_{jk}} \arrow[dl, "\pi_j'"'] \arrow[dr, "\pi_k'"] & \\
 & p_j \arrow[ul, "\delta_{ij}"] \arrow[rr, "\delta_{jk}"] & & p_k
\end{tikzcd}$$

The case of δ_{ik} is just similar and omitted. ■

In our context, the construction in Proposition 3.6 gives us the following higher-order correspondence:

$$\langle p_1,\ \delta',\ \langle p_2,\ \delta_{23},\ p_3\rangle\rangle. \tag{19}$$

with $\delta' \stackrel{\text{def}}{=} (\pi_2'^{-1} \circ \delta_{12}) \cup (\pi_3'^{-1} \circ \delta_{13})$. Thus once canonical binary projections from δ_{23} is calculated, we soon get the desired higher-order correspondence. A similar procedure applies to form a higher-order correspondence of a form (4) of Introduction.

3.6 Conclusion. Note that two procedures we discussed above easily generalise to the case when n is arbitrary. Since the decomposition-composition procedures provide a method of translation between different forms of higher-order correspondences such as (4) and (5), we have now answered the question posed in Introduction, both abstractly by our proof of Theorem 1.1 in Section 2, and concretely by the discussions we have just given in the present chapter.

References

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[2] Honda, K. Notes on P-Algebra (2): Group Presentation of Process Structure. A typescript. March 1995. available at ftp.cs.man.ac.uk.