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Notes on P-Algebra (4): Algebra over Process Structure, Part I.

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Abstract

We develop theory of algebra over processes, based on an abstract treatment of process structure. The abstract framework of process structures is developed (Part I), and, on its basis, theory of algebra is developed where the basic results including Birkhoff-like Variety Theorem are proved (Part II). We also discuss properties of significant concrete examples of algebras, taken from extant theories of concurrency. Part I develops the abstract theory of process structure encompassing a wide range of structures for process theory including concrete structures we have discussed in the preceding notes. The theory is used as foundations of the algebraic development in Part II, and is interesting in its own right.

1 Introduction

The aim of the present notes is two-fold. The first aim is to develop a notion of algebra over process structures together with an appropriate system of notations. The abstract framework of the theory of algebra fairly follows the usual categorical treatment of "algebra in a category" with the category of process structure and p-maps as the underlying category. The new points however lie in the concrete representation of general notions like homomorphisms in the setting of process structure, as well as basic differences in several constructions such as quotient and other constructions. Such differences necessitate a new notation for presenting algebraic notions. The theory, which treats static aspect of manipulation of pure processes, will become the basis of treatment of process dynamics, to be developed in the subsequent notes.

The second aim concerns the framework in which these notions and results are presented. We develop theory of algebra in an abstract framework unlike in the preceding Notes [7, 8, 9]. Thus, rather than relying on concrete construction of permutation groups and partial injections over finite sets, we base our development on a pair of categories conforming to certain conditions, by which all essential results and constructions of the preceding notes are guaranteed. For the development of theory of algebra, this results in clarity of presentation of the notions and the proofs. The approach is also important when we need to extend the notion of processes beyond what has been studied in the preceding study of concurrency so far, cf. Examples 2.19. Our primary purpose in this paper however lies in exploitation of the abstract framework for lucid presentation, leaving deeper study of such possibility elsewhere.

The notes are divided into two parts. Part I gives abstract theory of process structure, which will be used in Part II and also in the subsequent Notes. The treatment of the theory is self-contained. Part II is devoted to the development of elementary theory of algebra. In the rest of Part I, Section 2 presents the foundational structures of the abstract theory called *Connection Domains*, which gives a relational algebra of connections among processes in a categorical framework. Section 3 then studies an abstract theory of process structure based on connection domains. By varying the underlying connection domain, quite different kinds of "process structures" arise (including the familiar notion of sets themselves), each of which are shown to induce a set-like theory as we developed in the preceding Notes. All the main results of Notes 1 and 3 are established in this abstract setting, purely relying on the abstract algebra of connections. Surprisingly, the generality results in much shorter and more lucid proofs than those of the preceding Notes.

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2 Connection Domain

In functions or relations over a set, some elements of a set are related to some elements of a set. The way to relate an element to another element is quite simple: we just take a tuple of two elements. In the theory of processes, this simple scheme is no longer valid. Two processes (corresponding to two elements) may be related in diverse ways, considering how interface points of processes are connected to each other. This was implicit in the study of process algebra [11, 6], embodied in the notion of free names, or study of composita and its generalisations [2, 3, 13], embodied in the notion of "variables" (of, say, open terms), and is made explicit by the study of process structure [7], as well as by such works as [1, 5, 12]. Specifically what [7] showed is that a coherent theory of maps and relations, with which we manipulate processes collectively, can be constructed based on such notions, just like a set theory is a convenient means to manipulate a collection of elements. In this section and the next, we develop an abstract framework which allows such development. We can list relational algebra in [4] as a predecessor of the theory in this section, though the orientation and certain essential technicalities are quite different.

We start from a pair of categories which underlies the universe of processes.

2.1 Definition. (connection predomain) A *connection predomain* is a pair of locally small categories $(\mathcal{B}, \mathcal{C})$, where:

(i) In $\mathcal{B}$, all arrows are automorphisms. Objects are called *proto-processes*, ranged over by $p, q, \ldots$, and the set of automorphisms of an object p is called the *symmetries of* p, written $\mathcal{S}(p)$, denoted by $\rho, \rho', \ldots$. We assume there is a faithful functor from $\mathcal{B}$ to $\mathcal{C}$ which is surjective on objects. Thus we safely identify objects of $\mathcal{C}$ and those of $\mathcal{B}$. Note symmetries are (perhaps properly) contained in the automorphisms in $\mathcal{C}$.

(ii) The arrows of $\mathcal{C}$ are often called *connections*, ranged over by $\delta, \delta', \ldots$. On each homset, we assume a bijective operation $(\ )^\bullet : hom(p,q) \rightarrow hom(q,p)$ such that:

(1) $$\delta^{\bullet\bullet} = \delta \qquad (\delta_2 \circ \delta_1)^\bullet = \delta_1^\bullet \circ \delta_2^\bullet \qquad \delta \circ \delta^\bullet \circ \delta \ = \ \delta,$$

and, moreover, if δ is an isomorphism then $\delta^\bullet = \delta^{-1}$. We call $\delta^\bullet$ the *semi-inverse* of δ.

(iii) An endo-morphism δ over some p satisfying $\delta \circ \delta = \delta$ (idempotent) and $\delta^\bullet = \delta$ (hermetian), is called a *semi-unit on* p, denoted by $u, v, \ldots$. For any semi-units u, v in the same homset, we assume $u \circ v = v \circ u$ always hold.

A connection pre-domain is *pointed* if $\mathcal{C}$ has a terminator.

Often we denote a connection predomain by its second component, i.e. its category of connections, together with the specification of symmetries, from which we can recover the original pair. This is what we mean when we say: "In a connection predomain $\mathcal{C}$" etc. We write $\mathbf{1}_p$ for the identity on p, or $\mathbf{1}$ if no specific object is of interest to us. Note $\mathbf{1}^\bullet = \mathbf{1}$ since $\mathbf{1}^{-1} = \mathbf{1}$. This also shows that $\mathbf{1}$ is a semi-unit, an easy but an essential fact. Later we shall see $\mathbf{1}$ is, in a suitable sense of the term, the *maximum* among the set of semi-units on a proto-process. We also note:

$$u, v \text{ semi-units} \;\; \Rightarrow \;\; u \circ v \text{ a semi-unit} \tag{2}$$

which is precisely because of commutativity, a fact from elementary operator algebra.

The role of the notions and equations in Definition 2.1 would become clear when they are set to real use, which is soon. In brief: a proto-process is conceived as a geometric object which is specified by its symmetries and possible connections with other objects as given in $\mathcal{C}$. Semi-units are essentially partial identities, specifying "part of p". Then the identity on p is the "whole of p". All these comments are to be made precise later. In the following discussions, we fix some connection predomain $\mathcal{C}$.

First we introduce a basic equivalence on a homset. Two connections connecting the same pair of proto-processes may as well be regarded as essentially the same if the they are mediated by symmetries.

2.2 Definition. (equivalence) Given $\delta, \delta' \in hom(p_1, p_2)$, we write $\delta \sim \delta'$ iff $\delta' = \rho_2 \circ \delta \circ \rho_1$, where $\rho_1 \in \mathcal{S}(p_1)$ and $\rho_2 \in \mathcal{S}(p_2)$. Then δ and δ' are *equivalent.*

One can readily verify $\sim$ is indeed an equivalence relation on each homset. This equivalence relation is a staple when dealing with proto-processes and their connections. Definition 2.2 immediately tells us:

2.3 Proposition. *If $\delta' \sim \delta_2 \circ \delta_1$ then there exists $\delta'_1 \sim \delta_1, \delta'_2 \sim \delta_2$ such that $\delta' = \delta'_2 \circ \delta'_1$.*

Next, a connection with a special feature plays an essential role in the theory.

2.4 Definition. (map) A connection $\delta \in hom(p_1, p_2)$ is a *map* when: (a) it is *symmetry-preserving*, i.e. $\forall \rho_1 \in \mathcal{S}(p_1).\exists \rho_2 \in \mathcal{S}(p_2).\ \delta \circ \rho_1 = \rho_2 \circ \delta$, and: (b) it *covers the codomain*, i.e. $\delta \circ \delta^\bullet = \mathbf{1}_{p_2}$. We write $f, g, h, \ldots$ to denote maps.

For example, in a pointed predomain, the unique connection from any proto-process to a terminator is a map, as is easily verified. Another immediate example is symmetries of any proto-process. Symmetry-preservation in this setting is the same thing as the well-known fact that a symmetry as an element of a group induces an inner automorphism on the group. One may also find it interesting to see the difference in ways of deriving maps from arrows in a relational universe between here and in [4].

As may be expected, two maps compose to yield another map.

2.5 Proposition. *If $f : p \to q$ and $g : q \to r$ are maps, then $g \circ f$ is again a map.*

PROOF: First $g \circ f$ is is symmetry preserving because, for any $\rho \in \mathcal{S}(p)$, $g \circ f \circ \rho = g \circ \rho' \circ f = \rho'' \circ g \circ f$ where $\rho' \in \mathcal{S}(q)$ and $\rho'' \in \mathcal{S}(r)$. Second it covers the co-domain because: $(g \circ f) \circ (g \circ f)^\bullet = g \circ f \circ f^\bullet \circ g^\bullet = \mathbf{1}$. ■

A connection which is a map owns the following property in addition to Proposition 2.3, which may not hold for general connections.

2.6 Proposition. *Given two maps $f : p \to q$ and $g : q \to r$, if $f' \sim f$ and $g' \sim g$, we have $g \circ f \sim g' \circ f'$.*

PROOF: Put $f' = \rho_q \circ f \circ \rho_p$ and $g' = \rho_r \circ g \circ \rho'_q$, with ρ_p etc. denoting symmetries on appropriate objects. Then, by symmetry preservation, $g' \circ f' = \rho_r \circ g \circ \rho'_q \circ \rho_q \circ f \circ \rho_p = \rho''_r \circ g \circ f \circ \rho_p$ for some $\rho''_r \in \mathcal{S}(r)$, as required. ■

Now suppose f and $f^\bullet$ are both maps from p to q. Then they are necessarily isomorphisms [because: $f \circ f^\bullet = \mathbf{1}$ and $f^\bullet \circ f = (f \circ f^\bullet)^\bullet = \mathbf{1}$]. In that case, if $\delta \in hom(p,p)$, $f \circ \delta \circ f^\bullet \in \mathcal{S}(q)$ if and only if $\delta \in \mathcal{S}(p)$. Thus f and $f^\bullet$ induce a group isomorphism by conjugation between symmetries of two related objects. Conversely, isomorphisms with this property are always invertible maps. Such an isomorphism captures "essential sameness" of two proto-processes better than the ordinary isomorphisms in that it also reflects the structure of symmetries, and, thus, may deserve its own name.

2.7 Definition. (strong isomorphism) An isomorphism whose inverse and itself are both maps is called a *strong isomorphism.* Two objects equated by a strong isomorphism are *strongly isomorphic.*

For example, in a pointed predomain, two terminators are (not only isomorphic but also) strongly isomorphic. Also, symmetries are necessarily strong automorphisms. The converse may not hold in general. Note also strong isomorphisms compose to yield another strong isomorphism, by Proposition 2.5: therefore, if f is a strong isomorphism and $f \sim f'$, then, since symmetries are such, f' is also a strong isomorphism.

We can summarise the findings in terms of maps we have gotten so far by saying that, given a connection domain $\mathcal{C}$, its objects and maps form its subcategory, in which isomorphisms are precisely strong isomorphisms, and in which $\sim$ is always a congruence on morphisms, the properties being not in general true in $\mathcal{C}$.

The following two notions, defined using maps, play the key role in enabling certain set-like operations on processes, as treated in the next section.

2.8 Definition. (tabulation and representation)

(i) A *semi-tabulation of* δ is a pair of maps f, g (in this order) such that $\delta = g \circ f^\bullet$. Then a *tabulation* of δ is a semi-tabulation f, g of δ such that, for any semi-tabulation f', g' of δ, there is a unique map h such that $f' = f \circ h$ and $g' = g \circ h$. The common domain of a tabulation is called a *tabulating object* of the tabulation.

(ii) Given a proto-process p, $\mathcal{E} \subset hom(p,p)$ form a *pre-symmetry group* when: (a) it forms a group with $(\ )^\bullet$ as the inverse and $\circ$ as the product, and: (b) if $\rho, \rho' \in \mathcal{S}(p)$ and $\delta \in \mathcal{E}$ then $\rho \circ \delta \circ \rho' \in \mathcal{E}$. A *representation* of a pre-symmetry group $\mathcal{E}$ on p is a map $f : p \to q$ for some q such that: (a) $f^\bullet \circ f = id_\mathcal{E}$ ($id_\mathcal{E}$ is the identity of $\mathcal{E}$), and: (b) the function $\psi : \mathcal{E} \to hom(q,q)$ given by $\delta \mapsto f \circ \delta \circ f^\bullet$, is a bijection between $\mathcal{E}$ and $\mathcal{S}(q)$. q is then the *representing object.* An element of a pre-symmetry group is called a *pre-symmetry.*

Thus a (semi-) tabulation gives a way of expressing an arrow in $\mathcal{C}$ as an object in $\mathcal{C}$, while a representation realises a "part of an object" with possibly additional symmetries, as another object. An immediate example of the former is $\mathbf{1}_p$ and a map f which tabulate f itself. An example of the latter is $\mathcal{S}(p)$ for arbitrary p which is always represented as $\mathbf{1}_p$. Or, in a pointed predomain, the unique arrow of the terminal object is represented by the arrow itself. On the other hand, any map f is a representation of a presymmetry group, the latter being easily calculated as $\{f^\bullet \circ \rho \circ f \mid \rho \in \mathcal{S}(q)\}$ where q is the codomain of f. We note, in (ii), ψ needs only be surjective and then it is not only bijective but also a group isomorphism (by conjugation), since if $f \circ \delta \circ f^\bullet = f \circ \delta' \circ f^\bullet$ then $\delta = \delta'$ [using $f^\bullet \circ f = e_\mathcal{E}$ ($e_\mathcal{E}$ being the identity of the presymmetry group) and $f \circ f^\bullet = \mathbf{1}_q$], and it is easy to check ψ is a group homomorphism.

The following shows that two tabulations of a connections are always mediated by strong isomorphisms, similarly for two representations of a presymmetry group.

2.9 Proposition.

(i) *Given a connection δ, suppose f, g with the common domain q and f', g' with the common domain r both tabulate δ. Then there is a unique strong isomorphism $h : r \to q$ such that $f' = f \circ h$ and $g' = g \circ h$. Conversely, if f, g tabulates δ with the tabulating object q and there is a strong isomorphism h from q to r, then $f \circ h, g \circ h$ also tabulates δ.*

(ii) *Given a pre-symmetry group $\mathcal{E}$ on p, if it has two representations i and i', then $i' \circ i^\bullet$ is a strong isomorphism mediating two representing objects. Conversely, if i is a representation of a pre-symmetry group with r being the representing object, and there is a strong isomorphism h with domain r, then $h \circ i$ is also a representation of the same pre-symmetry group.*

PROOF:

(i) Standard, noting invertible maps are strong iso's.

(ii) Let the two representations be $f : p \to q$ and $g : p \to r$. We show $g \circ f^\bullet$ and $f \circ g^\bullet$ are both maps, which proves the first half of the statement. By definition, $g \circ f^\bullet$ induces a group isomorphism from $\mathcal{S}(q)$ to $\mathcal{S}(r)$ via $\mathcal{E}$, in the way:

$$\rho_q \mapsto f^\bullet \circ \rho_q \circ f \mapsto g \circ f^\bullet \circ \rho_q \circ f \circ g^\bullet$$

which shows that it is symmetry-preserving in both ways. Next because $g^\bullet \circ g$ is the identity of $\mathcal{E}$ (by definition of representation) and $f \circ f^\bullet = \mathbf{1}$ (since f is a map), we get $f \circ g^\bullet \circ g \circ f^\bullet = \mathbf{1}$, which shows they are strong isomorphisms. The second statement is easy. ■

In view of the above (i), we shall often write $\underline{\delta}$ for the tabulating object of a selected tabulation of δ, and speak of "the" tabulation of δ (thus invoking the axiom of choice). Similarly, from now on, we shall often speak of "the" representation of a presymmetry group, by (ii) above.

In various situations, we need to deal with fine structures of, and operations on, connections. Imagine a connection as a *tube* connecting one geometric object to another: thus, in general, there may be multiple tubes for connecting two given objects. These multiple tubes can be related (apart from equivalence on them we have introduced already) in the following two basic ways.

2.10 Definition. (coherence and inclusion)

(i) On each homset, we define a binary relation $\asymp$, the *coherence relation*, so that

$$\delta_1 \asymp \delta_2 \quad \stackrel{\text{def}}{\Leftrightarrow} \quad \delta_1^\bullet \circ \delta_2 \text{ and } \delta_2 \circ \delta_1^\bullet \text{ are both semi-units.}$$

Then we say δ_1 and δ_2 are (mutually) *coherent*.

(ii) Again on each homset, we define a binary relation $\leq$, the *inclusion ordering*, so that:

$$\delta_1 \leq \delta_2 \quad \stackrel{\text{def}}{\Leftrightarrow} \quad \delta_2 \circ \delta_1^\bullet \circ \delta_2 = \delta_1 \ \wedge \ \delta_1 \circ \delta_2^\bullet \circ \delta_1 = \delta_1.$$

Then we say δ_1 is *contained in* δ_2.

Note, in (i), if $\delta_1 \asymp \delta_2$ then $\delta_1^\bullet \circ \delta_2 = \delta_2^\bullet \circ \delta_1$ since the semi-inverse of a semi-unit is itself. The following is useful when studying the inclusion order.

2.11 Lemma. *Suppose $\delta_1 \leq \delta_2$. Then if one of i, j, k is 1 and others being in the set $\{1, 2\}$, we always have: $\delta_i \circ \delta_j^\bullet \circ \delta_k = \delta_1$.*

PROOF: Easy calculation. For example, $\delta_1 \circ \delta_1^\bullet \circ \delta_2 = \delta_1 \circ \delta_1^\bullet \circ \delta_2 \circ \delta_1^\bullet \circ \delta_2 = \delta_1 \circ \delta_1^\bullet \circ \delta_1 = \delta_1$, the last step using (1) of Definition 2.1. Others are similar. ■

Using this, we now state the basic facts about two relations we have just introduced.

2.12 Proposition.

(i) *$\asymp$ is symmetric and reflexive. $\leq$ is reflexive, transitive and anti-symmetric, i.e. is a partial order.*

(ii) *If $\delta_1 \leq \delta_2$ then $\delta_1 \asymp \delta_2$. Also, $\delta_1 \leq \delta_1' \asymp \delta_2' \geq \delta_2$ implies $\delta_1 \asymp \delta_2$.*

(iii) *Both $\leq$ and $\asymp$ commute with $(\ )^\bullet$ and $\circ$. If $\delta_1 \sim \delta_1'$ and $\delta_1 \leq \delta_2$ (resp. $\delta_2 \leq \delta_1$, resp. $\delta_1 \asymp \delta_2$), then there exists δ_2' such that $\delta_1' \leq \delta_2'$ (resp. $\delta_2' \leq \delta_1'$, resp. $\delta_1' \asymp \delta_2'$).*

PROOF:

(i) By definition, $\asymp$ is clearly symmetric, and its reflexivity is also easy to calculate. Regarding $\leq$, since $\delta \circ \delta^\bullet \circ \delta = \delta$ by axiom, $\leq$ is indeed reflexive. Next suppose $\delta_1 \leq \delta_2$ and $\delta_2 \leq \delta_3$ both hold. Then, by Lemma 2.11,

$$\begin{aligned}\delta_3 \circ \delta_1^\bullet \circ \delta_3 &= \delta_3 \circ \delta_2^\bullet \circ \delta_1 \circ \delta_2^\bullet \circ \delta_3 \\ &= (\delta_3 \circ \delta_2^\bullet \circ \delta_2) \circ \delta_1^\bullet \circ (\delta_2 \circ \delta_2^\bullet \circ \delta_3) \\ &= \delta_2 \circ \delta_1^\bullet \circ \delta_2 \\ &= \delta_1,\end{aligned}$$

which means $\delta_1 \leq \delta_3$, so that $\leq$ is transitive. Finally suppose $\delta_1 \leq \delta_2$ and $\delta_2 \leq \delta_1$. Again by Lemma 2.11,

$$\delta_1 \ = \ \delta_2 \circ \delta_1^\bullet \circ \delta_2 \ = \ \delta_2$$

hence done.

(ii) The first statement says that $\delta_1 \leq \delta_2$ implies both $\delta_1 \circ \delta_2^\bullet$ and $\delta_1^\bullet \circ \delta_2$ being semi-units, which is an easy calculation. For example,

$$\begin{aligned}(\delta_1 \circ \delta_2^\bullet)^\bullet &= \delta_2 \circ \delta_1^\bullet && \text{(by definition)} \\ &= \delta_2 \circ \delta_2^\bullet \circ \delta_1 \circ \delta_2^\bullet && \text{(by } \delta_1 \leq \delta_2\text{)} \\ &= \delta_1 \circ \delta_2^\bullet && \text{(by Lemma 2.11).}\end{aligned}$$

Others are similar. For the second statement, assume $\delta_1 \leq \delta_1' \asymp \delta_2' \geq \delta_2$. Then, by Lemma 2.11,

$$\delta_1 \circ \delta_2^\bullet = \delta_1 \circ \delta_1^\bullet \circ \delta_1' \circ \delta_2' \circ \delta_2^\bullet \circ \delta_2 = (\delta_1 \circ \delta_1^\bullet) \circ (\delta_1' \circ \delta_2') \circ (\delta_2^\bullet \circ \delta_2).$$

Observing all parenthesised expressions are semi-units (by the reflexivity of $\asymp$ and by assumption), since semi-units are composable as noted in (2), the whole is a semi-unit. Similarly for $\delta_2^\bullet \circ \delta_1$, hence done.

(iii) We only show the case for $\leq$. The case for $\asymp$ can be shown with a similar reasoning. Assume $\delta_1 \leq \delta_2$. We show $\delta_1^\bullet \leq \delta_2^\bullet$. Indeed, for example,

$$\delta_2^\bullet \circ \delta_1 \circ \delta_2^\bullet \ = \ (\delta_2 \circ \delta_1^\bullet \circ \delta_2)^\bullet \ = \ \delta_1^\bullet,$$

similarly for the other case. For $\circ$, let $\delta_1 \leq \delta_2$. We show $\delta \circ \delta_1 \leq \delta \circ \delta_2$ for any composable δ. Assume $\delta_1 \leq \delta_2$, i.e. $\delta_2 \circ \delta_1^\bullet \circ \delta_2 = \delta_1$. Then:

$$\begin{aligned}
\delta \circ \delta_2 \circ (\delta \circ \delta_1)^\bullet \circ \delta \circ \delta_2 &= \delta \circ \delta_2 \circ \delta_1^\bullet \circ \delta^\bullet \circ \delta \circ \delta_2 \\
&= \delta \circ \delta_2 \circ \delta_2^\bullet \circ \delta_1 \circ \delta_2^\bullet \circ \delta^\bullet \circ \delta \circ \delta_2 \\
&= \delta \circ (\delta_2 \circ \delta_2^\bullet) \circ (\delta_1 \circ \delta_2^\bullet) \circ (\delta^\bullet \circ \delta) \circ \delta_2.
\end{aligned}$$

Observing all parenthesised expressions above are semi-units (using the above (ii) for the middle one), we can use commutativity of their composition. Thus we get $(\delta \circ \delta^\bullet \circ \delta) \circ (\delta_2 \circ \delta_2^\bullet \circ \delta_1 \circ \delta_2^\bullet \circ \delta_2)$ which is nothing but $\delta \circ \delta_1$ (use the axiom for the first expression, Lemma 2.11 for the second), as required. Similarly we can show $\delta_1 \circ \delta_2$ implies $\delta_1 \circ \delta \leq \delta_2 \circ \delta$, with which we have now shown the required result. The second statement then follows easily from the first statement. ■

Given this, let us study basic facts relating semi-units and inclusion/coherence, which often become essential in our subsequent discussions.

2.13 Proposition.

(i) *$u \in hom(p,p)$ is a semi-unit if and only if $u \leq \mathbf{1}_p$.*

(ii) *If u, v are semi-units on p and q respectively, then for any $\delta : p \to q$ we have $v \circ \delta \circ u \leq \delta$.*

(iii) *If u, v are semi-units on the same object, then $u \asymp v$ always. Moreover we have $u \leq v$ if and only if $u \circ v (= v \circ u) = u$. Indeed $u \circ v = v \circ u$ gives the meet of u and v with respect to $\leq$.*

(iv) *We generalise* (iii) *above as follows. Let $\delta_1 \asymp \delta_2$. Let $u = \delta_1^\bullet \circ \delta_2$ and $v = \delta_1 \circ \delta_2^\bullet$. Then $\delta_1 \circ u = \delta_2 \circ u = v \circ \delta_1 = v \circ \delta_2$ and, moreover, this connection gives the meet of δ_1 and δ_2 with respect to $\leq$.*

(v) *If a connection predomain is pointed, each $hom(p,q)$ has the minimum element. Indeed, $!_q^\bullet \circ !_p$ gives such, where $!_p$ is the unique arrow from p to a selected terminator. The minimum element in $hom(p,q)$ constructed in this way is denoted $\mathbf{0}_{p\to q}$. We write $\mathbf{0}_p$ to denote $\mathbf{0}_{p\to p}$, if such exists. Note $\mathbf{0}_p$ is clearly a semi-unit.*

Proof: (i) is immediate from the definition of semi-units. Using this and also Proposition 2.12 (iii) we know (ii) holds. For (iii) the first statement is obvious. The second is also easy by by expanding the conditions of the left hand side and using commutativity of semi-units and Lemma 2.11. Now (ii) above tells us $u \circ v$ is a lower bound of $\{u, v\}$. Let w be another lower bound. By (i) w is also a semi-unit. But $w \circ (u \circ v)^\bullet \circ w = w \circ u \circ w \circ v$ which is equal to w by the preceding result, similarly $u \circ v \circ w^\bullet \circ u \circ v = w$, which shows $u \circ v \leq w$, as required. (iv) is similar to (iii). (v) is an easy reasoning using the uniqueness of terminator. ■

Deeper understanding of connections is gained by considering special semi-units associated with each connection. We remember, by Proposition 2.12 (ii), that, for any connection $\delta : p \to q$, we have $\delta \asymp \delta$, i.e. $\delta^\bullet \circ \delta : p \to p$ and $\delta \circ \delta^\bullet : q \to q$ are both semi-units.

2.14 Definition. (sections) The *preimage* (or sometimes the *source section*) of δ is a semi-unit $\delta^\bullet \circ \delta$, denoted $\mathtt{pre}(\delta)$. The *image* (or sometimes the *target section*) of δ is a semi-unit $\delta \circ \delta^\bullet$, denoted $\mathtt{im}(\delta)$.

With the analogy of a tube, one may regard a section as that part of the object connected to a tube. The following properties tells us that this analogy is not far from the reality.

2.15 Proposition.

(i) $\delta \circ \mathtt{pre}(\delta) = \mathtt{im}(\delta) \circ \delta = \delta$. Moreover, if u is a semi-unit s.t. $\delta \circ u = \delta$ (resp. $u \circ \delta = \delta$), then $\mathtt{pre}(\delta) \leq u$ (resp. $\mathtt{im}(\delta) \leq u$).

(ii) *If* $\delta_1 \leq \delta_2$ *then* $\mathtt{pre}(\delta_1) \leq \mathtt{pre}(\delta_2)$ *and* $\mathtt{im}(\delta_1) \leq \mathtt{im}(\delta_2)$.

(iii) *If* $\delta_1 \asymp \delta_2$ *then we have*: $\mathtt{pre}(\delta_1) \leq \mathtt{pre}(\delta_2)$ *iff* $\mathtt{im}(\delta_1) \leq \mathtt{im}(\delta_2)$ *iff* $\delta_1 \leq \delta_2$.

(iv) *Given* $\delta : p \to q$, *if either* $\mathtt{pre}(\delta) = \mathbf{0}_p$ *or* $\mathtt{pre}(\delta) = \mathbf{0}_q$ *then* $\delta = \mathbf{0}_{p \to q}$.

(v) *Two connections* $\delta_1, \delta_2 : p \to q$ *are* disjoint, *written* $\delta_1 // \delta_2$ *if* $\mathtt{pre}(\delta_1) \wedge \mathtt{pre}(\delta_2) = \mathbf{0}_p$ *and* $\mathtt{im}(\delta_1) \wedge \mathtt{im}(\delta_2) = \mathbf{0}_q$. *Then if* $\delta_1 // \delta_2$ *we have* $\delta_1 \asymp \delta_2$.

(vi) (inversion lemma) *If* $f_2 = g \circ f_1^\bullet$ *and* $\mathtt{im}(f_2) = \mathbf{1}$, *then* $g = f_2 \circ f_1$.

Proof: The first half of (i) is by the last equation in (1) of Definition 2.1. For the second half, suppose $\delta \circ u = \delta$. Then by multiplying by $\delta^\bullet$ from the left, we have $\mathtt{pre}(\delta) \circ u = \mathtt{pre}(\delta)$, which, by (iii) of Proposition 2.13, is equivalent to the required ordering. Similarly for the case of the target section, hence done. (ii) is obtained by computing $\mathtt{pre}(\delta_2) \circ (\mathtt{pre}(\delta_1))^\bullet \circ \mathtt{pre}(\delta_2)$ and $\mathtt{pre}(\delta_1) \circ (\mathtt{pre}(\delta_2))^\bullet \circ \mathtt{pre}(\delta_1)$ by Lemma 2.11. For (iii), assume $\delta_1 \asymp \delta_2$ and $\mathtt{pre}(\delta_1) \leq \mathtt{pre}(\delta_2)$. Then we have the following series of equations to reach one of the defining equations for $\delta_1 \leq \delta_2$:

$$
\begin{array}{rcll}
\delta_1^\bullet \circ \delta_1 \circ \delta_2^\bullet \circ \delta_2 & = & \delta_1^\bullet \circ \delta_1 & \text{Prop. 2.13 (iii)} \\
\delta_1 \circ \delta_1^\bullet \circ \delta_1 \circ \delta_2^\bullet \circ \delta_2 & = & \delta_1 & (1) \\
(\delta_1 \circ \delta_2^\bullet) \circ (\delta_1 \circ \delta_1^\bullet) \circ \delta_2 & = & \delta_1 & \text{s.u. commutes} \\
(\delta_2 \circ \delta_1^\bullet) \circ (\delta_1 \circ \delta_1^\bullet) \circ \delta_2 & = & \delta_1 & \text{s.u. invertible} \\
\delta_2 \circ \delta_1^\bullet \circ \delta_2 & = & \delta_1. & \text{Lemma 2.11}
\end{array}
$$

Another equation, i.e. $\delta_1 \circ \delta_2^\bullet \circ \delta_1 = \delta_1$, can be derived from $\delta_2^\bullet \circ \delta_1 \circ \delta_2^\bullet \circ \delta_1 = \delta_2^\bullet \circ \delta_1$ in a similar way. Then use (ii) above to get the whole logical equivalences. Regarding (iv), let $\delta : p \to q$ and $\mathtt{im}(\delta) = \mathbf{0}_q$. By (i) we know $\delta \circ \mathbf{0}_q = \delta$. By uniqueness of terminator, immediately $\delta = \mathbf{0}_{p \to q}$. If $\mathtt{pre}(\delta) = \mathbf{0}_p$, the same reasoning gives $\delta^\bullet = \mathbf{0}_{q \to p}$. But $\leq$ commutes with $(\ )^\bullet$, therefore δ is also the minimum in $hom(p, q)$, hence should be $\mathrm{p}\mathbf{0}_{p\to q}$, as required. (v) is immediate by noting $\mathbf{0}$ itself is a semi-unit. For (vi) notice $f_2 \circ f_1 = g \circ f_1^\bullet \circ f_1 \leq g$ by Proposition 2.13 (i), while $g = (f_2 \circ f_2^\bullet) \circ g = f_2 \circ f_1 \circ g^\bullet \circ g \leq f_2 \circ f_1$. Then use Proposition 2.12 (ii). ■

It is notable that sections often appeared in our preceding discussions without being named as such. Indeed, they do appear already in the third equation of 1 of Definition 2.1: Or a map is nothing but a connection for which, beside symmetry preserving, the target section is identity: Further, in a representation, we require its source section to be the group identity of the presymmetry group. One may say that a tube is, before anything, specified by the parts of processes its two ends are attached to.

We now push the analogy of a tube further. Let us have two mutually coherent connections (imagine two tubes connecting two objects whose intersecting part again forms a tube). Then we have the *meet* of the two (written $\delta \wedge \delta'$) by Proposition 2.13 (iv) above, which is nothing but the intersecting part itself. We may as well have, then, the *join* (written $\delta \vee \delta'$), which is the result of adding two connections. The third operation between two coherent connections would be to take *difference* between the two. Since a homset in $\mathcal{C}$ may not form a boolean lattice in general, however, the idea cannot simply follow from the general notion of complements. We thus formulate the notion below for our specific case. We only define the notion when a connection is included in another: existence of join would cater for a more general case.

2.16 Definition. (difference) Suppose $\delta_1 \leq \delta_2 : p \to q$ and $\mathbf{0}_{p\to q}$ exists. Then a *difference between δ_2 and δ_1* (in this order) is a connection, say δ', in the same homset such that:

(i) $\delta' \leq \delta_2$.
(ii) $\delta' \wedge \delta_1 = \mathbf{0}_{p\to q}$ (note that the meet exists by (v) of Proposition 2.13).
(iii) $\delta' \vee \delta_1$ exists and is equal to δ_2.

One should note that a difference may not be determined uniquely. The following becomes important to ameliorate the situation.

2.17 Proposition. *Let $\delta_1 \leq \delta_2$. If δ' is a difference between δ_2 and δ_1, then* $\mathtt{pre}(\delta')$ *is a difference between* $\mathtt{pre}(\delta_2)$ *and* $\mathtt{pre}(\delta_1)$*, similarly for* $\mathtt{im}(\delta')$*. Moreover if δ'' is also a difference from δ_2 by δ_1 and if* $\mathtt{pre}(\delta') = \mathtt{pre}(\delta'')$ *or* $\mathtt{im}(\delta') = \mathtt{im}(\delta'')$ *holds, then $\delta' = \delta''$.*

Conversely one can construct a difference of connections from that of their sections, as can be easily seen.

We have almost paved the way towards *connection domain*, a special kind of connection predomain which is the basis of various theoretical developments from now on. Essentially speaking, a connection domain is a connection predomain where all desirable constructions we have seen so far can be performed. The definition follows.

2.18 Definition. (connection domain) A *connection domain* is a pointed connection predomain such that:

(i) All connections have semi-tabulations;

(ii) All presymmetry groups have representations; and

(iii) For any p, q, if $\{\delta_i\}_{i \in I} \subset hom(p, q)$ and, moreover, $\delta_i \asymp \delta_j$ for each $i, j \in I$, then always $\wedge\{\delta_i\}$ and $\vee\{\delta_i\}$ exist, which moreover commute with $(\ )^{-1}$ and $\circ$. Similarly if $\delta \leq \delta'$ then their difference always exists.

Note that commutativity of infima/suprema with respect to $\circ$ and $(\ )^\bullet$ is well-defined because $\asymp$ commutes with these operations. Note also all homsets in a connection domain has the minimum element, by Proposition 2.13 (v). Before showing a few essential properties of connection domains, we give some examples.

2.19 Examples. (connection domains)

(i) The first example is the one-object, one-arrow category. Here the unique arrow is the only symmetry hence a map. We can check that, indeed, a pair of this unique arrow semi-tabulates that arrow itself, and the (only) pre-symmetry group is represented by the unique object, which is also a terminator. Suprema/infima conditions are trivial. This connection domain is called Elm.

(ii) The second example is from our preceding study. Take (concrete) permutation groups on finite sets as proto-processes [here a permutation group is a set of bijections over a set forming a group; note there is a permutation group on an empty set consisting of a sole element, an empty map]. Their symmetries are precisely those bijective actions. Then a connection domain is given by those proto-processes together with partial one-one map between the underlying sets. Infimum and supremum are taken simply as set intersection and join. This domain is called $\mathsf{FinPerm}$. If we allow infinite sets, then the domain is called Perm.

(iii) An important ramification of the above is a theory of webs. With the same set of proto-processes as $\mathsf{FinPerm}$ (or, more interestingly, Perm) with the same symmetries, we take connections as "webs", i.e. a relation R between two finite sets such that $R \circ R^\bullet \circ R = R$ (note this is precisely the last axiom in (1)). We leave it to the reader to compute such notions as maps, semi-tabulations, semi-units, etc.

(iv) Another ramification can be the set of one-dimensional simplical objects, i.e. graphs. They are given symmetries as graph automorphisms. Connections are, for example, any partial injection between their vertices. Tabulations are given by identifying related vertices. Representations are maximum subgraph making the presymmetries symmetries. This connection domain is called Grf.

(v) Take the collection of topological spaces as proto-processes. First let us take the auto-homeomorphisms as symmetries. Connections would again be given by partial injections. A (semi-)tabulation is given by identification topology. However this is not closed under representations: for this to be satisfied, we should take a pair of a topological space and a subgroup of auto homeomorphisms as a proto-process. If we take all such pairs, the resulting category is indeed a connection domain called Top.

Let us study a few fundamental properties of connection domains. First, the following may be said to crystallise the conditions on join, meet and differences in Definition 2.18, and is crucial for various applications of the theory.

2.20 Proposition. *In a connection domain, the set of semi-units on a given object forms a complete boolean lattice with respect to the inclusion ordering.*

PROOF: By (v) of Proposition 2.13, semi-units on the same object are always mutually coherent, hence, by Definition 2.18, it has joins and meets for all non-empty subset. Moreover by (iii) of Proposition 2.13, it has the top element $\mathbf{1}$, and since the connection domain has a terminator, (ii) tells us it has the bottom element $\mathbf{0}$. For distributivity we have, for example:

$$u \wedge (v \vee w) = u \circ (v \vee w) = (u \circ v) \vee (u \circ w) = (u \wedge v) \vee (u \wedge w).$$

Since the complement of a semi-unit are nothing but the difference between $\mathbf{1}$ and the semi-unit, we are done. ∎

As an easy but basic consequence, we have:

2.21 Corollary. *In a connection domain, a difference between two connections is uniquely determined. The difference between δ_2 and δ_1, uniquely determined by Corollary above, is written $\delta_2 \backslash \delta_1$.*

PROOF: By Propositions 2.17 and 2.20. ∎

Next we establish a couple of significant properties of connection domains concerning semi-tabulations and tabulations. Indeed we give the results for their generalised versions for a family of processes, which we first introduce below.

2.22 Definition. (generalised connections, semi-tabulations, and tabulations)

(i) In $\mathcal{C}$, assume given a family of proto-processes, $\{\delta_i\}_{i \in I}$. Then *connection over* $\{p_i\}_{i \in I}$ is a family of connections $\{\delta_{ij}\}_{i,j \in I}$ such that:

1. $\delta_{ij} : p_i \to p_j$ for each i, j.
2. $\delta_{ii} = \mathbf{1}_p$ for each $i \in I$. Moreover $\delta_{ij} = \delta_{ji}^{\bullet}$ for each i, j.

3. $\delta_{jk} \circ \delta_{ij} \leq \delta_{ik}$ for each i, j, k.

We often write $\langle \delta_{ij} \rangle$ to show $\{\delta_{ij}\}$ is a connection. We also define:

$$\langle \delta_{ij} \rangle \sim \langle \delta'_{ij} \rangle \overset{\text{def}}{\Leftrightarrow} \exists \{\rho_i \in \mathcal{S}(p_i)\}_{i \in I}.\ \delta'_{ij} = \rho_j^\bullet \circ \delta_{ij} \circ \rho_i.$$

Then $\langle \delta_{ij} \rangle$ and $\langle \delta'_{ij} \rangle$ are *equivalent*.

(ii) Given a connection $\langle \delta_{ij} \rangle$ over $\{p_i\}$, its *semi-tabulation* is a proto-process q and a family of maps $\{f : q \to p_i\}$ such that, for each i, j, we have $\delta_{ij} = f_j \circ f_i^\bullet$. Then a *tabulation* of $\langle \delta_{ij} \rangle$ is a semi-tabulation of $\langle \delta_{ij} \rangle$, say q and $\{f_i : q \to p_i\}$, such that for any other semi-tabulation q' and $\{f'_i : q' \to p_i\}$ of $\langle \delta_{ij} \rangle$, there is the unique map $g : q' \to q$ such that $f'_i = f_i \circ g$ for each i.

2.23 Convention and Notation. When I is finite, we shall always assume $I = 1, 2, .., n$ without any loss of generality, in which case we have an *n-ary connection.* Note the 0-*ary connection* is just an empty set. We may write $\langle \delta_{ij} : p_1, \ldots, p_n \rangle$ for an n-ary connection over $p_1, \ldots, p_n$.

We observe that the case of a binary connection is exactly the notion of connection we have been discussing so far. Thus we can consistently call generalised connections above simply *connections*. As to (ii) in Definition 2.22, note that any family of maps with the common domain always semi-tabulate some connection [put $\delta_{ij} = f_j \circ f_i^\bullet$ then $\delta_{ii} = f_i \circ f_i^\bullet = \mathbf{1}_{p_i}$, $\delta_{ji} = f_i \circ f_j^\bullet = \delta_{ij}^\bullet$ and $\delta_{jk} \circ \delta_{ij} = f_k \circ f_j{}^\bullet \circ f_j \circ f_i{}^\bullet \leq f_k \circ f_i{}^\bullet = \delta_{ij}$]. Note also that, by definition, *any* proto-process in $\mathcal{C}$ is a semi-tabulation of the 0-ary connection. Thus a tabulation for an 0-ary connection is nothing but the terminator in $\mathcal{C}$. Finally the standard reasoning tells us, if $\{f_i\}$ and $\{g_i\}$ both tabulate $\langle \delta_{ij} \rangle$, then there is the strong isomorphism such that, for each i, we have $f_i = g_i \circ h$.

The first result concerning semi-tabulation may not be so surprising.

2.24 Proposition. *In a connection domain, any n-ary connection has a semi-tabulation.*

PROOF: We use induction on n. If $n = 0$, take at least the terminal object in $\mathcal{C}$ [the cases for $n = 1, 2$ are also immediate, but we only need the case for $n = 0$: notice how the following inductive step works for $m = 0$]. Suppose the statement holds with $n = m$. Given $\langle \delta_{ij} : p_1, \ldots, p_n \rangle$, a $m+1$-ary connection on $p_1, \ldots, p_{m+1}$, let $\{\delta'_{ij}\}_{1 \leq i,j \leq m}$ be such that $\delta'_{ij} = \delta_{ij}$ for $1 \leq i, j \leq m$. Then $\{\delta'_{ij}\}$ is an m-ary connection on $\{p_1, \ldots, p_m\}$. By induction it has a semi-tabulation $\{\pi'_i : q' \to p_i\}$. Let $\xi_i = \delta_{im+1} \circ \pi'_{ij}$. Then we have:

$$\begin{aligned} \xi_j^\bullet \circ \xi_i &= \pi'^\bullet_j \circ \delta_{m+1j} \circ \delta_{im+1} \circ \pi'_i \\ &\leq \pi'^\bullet_j \circ \delta_{ij} \circ \pi'_i \\ &= \pi'_j \circ \pi'_j \circ \pi'^\bullet_i \circ \pi'_i \end{aligned}$$

which shows $\xi_j^\bullet \circ \xi_i$ is a semi-unit for any i, j similarly for $\xi_j \circ \xi_i^\bullet$, hence $\xi_i \asymp \xi_j$ for each i, j. Thus [since the join exists in connection domain], we can have $\vee\{\xi_i\} : q' \to p_{n+1}$.

Let a semi-tabulation of this connection be $f : q \to q'$ and $g = \pi_{m+1} : q \to p_{n+1}$. Put $\pi_i \stackrel{\text{def}}{=} \pi' \circ f$. Now we show $\{\pi_i\}_{1\le i\le m+1}$ semi-tabulates $\{\delta_{ij}\}_{1\le i,j\le m+1}$. Indeed, for $1 \le i, j \le m$, we have $\pi_j \circ \pi_i^\bullet = \pi_j \circ f \circ f^\bullet \circ \pi_i'^\bullet = \pi_j \circ \pi_i'^\bullet = \delta_{ij}$, while, if $1 \le i \le m$, we get

$$\begin{aligned}
\pi_{n+1} \circ \pi_i{}^\bullet &= g \circ f^\bullet \circ \pi_i'^\bullet \\
&= \vee\{\xi_j \circ \pi_i'^\bullet\} \\
&= \vee_{1\le j\le m}\{\delta_{jm+1} \circ \pi_j' \circ \pi_i'^\bullet\} \\
&= \vee_{1\le j\le m}\{\delta_{jm+1} \circ \delta_{ij}\} \quad = \quad \delta_{im+1}.
\end{aligned}$$

Finally $\pi_{m+1} \circ \pi_{m+1}^\bullet = \mathbf{1}_{p_{m+1}} = \delta_{m+1m+1}$, hence done. ■

A striking fact is that a connection domain has a tabulation whenever there is a semi-tabulation. This is one of the most basic facts about a connection domain, and has significant consequences as we shall see later. The proof reveals essential import of the notion of symmetries.

2.25 Proposition. (semi-tabulation means tabulation) *In a connection domain, if a connection (say $\langle\delta_{ij}\rangle$) has a semi-tabulation (say $\{f_i\}$), it also has a tabulation (say $\{g_i\}$), with the mediating map $\vee\{g_i \circ f_i^\bullet\}$. Specifically any n-ary connection has a tabulation. A semi-tabulation $\{f_i : r \to p_i\}$ of $\langle\delta_{ij}\rangle$ is a tabulation if and only if we have $\mathcal{S}(r) \stackrel{\text{def}}{=} \{\vee\{f_i^\bullet \circ \rho_i \circ f_i\} \mid \{\rho_i \in \mathcal{S}(p_i)\}$ satisfies $\forall i, j.\ \delta_{ij} = \rho_j \circ \delta_{ij} \circ \rho_i^\bullet\}$.*

Proof: Let $\langle\delta_{ij}\rangle_{i,j\in I}$ be a connection and $\{f_i : q \to p_i\}_{i\in I}$ be its semi-tabulation. We often omit the indexing set I since no confusion is possible. We take the following presymmetry group on q:

(i) Identity: the semi-unit $\vee\mathtt{pre}(f_i)$.

(ii) Elements: for each $\{\rho_i \in \mathcal{S}(p_i)\}$ such that $\delta_{ij} = \rho_j \circ \delta_{ij} \circ \rho_i^\bullet$ for each i, j, we take $\vee\{f_i^\bullet \circ \rho_i \circ f_i\}$.

We call this set $\mathcal{E}$. Let us check that $\mathcal{E}$ indeed forms a presymmetry group. Write τ_{f_i,ρ_i} for $f_i^\bullet \circ \rho_i \circ f_i$. First we note the join in (ii) above is well-defined because $\tau_{f_i,\rho_i} \asymp \tau_{f_j,\rho_j}$ is easy by calculations: $\tau_{f_i,\rho_i} \circ \tau_{f_j,\rho_j}^\bullet = f_i^\bullet \circ \rho_i \circ \delta_{ij}^\bullet \circ \rho_j \circ f_j = f_i^\bullet \circ \delta^\bullet \circ f_j = f_i^\bullet \circ f_i \circ f_j^\bullet \circ f_j$, but the last expression is indeed a semi-unit, similarly for another composition. Next we show they indeed forms a group with identity $\vee\mathtt{pre}(f_i)$. For closure under group operations, we first note that the set $\{\ \{\rho_i\} \mid \forall i, j \in I.\ \delta_{ij} = \rho_j \circ \delta_{ij} \circ \rho_i^\bullet\}$ itself forms a group in terms of pointwise composition and inverse, an easy verification. Next we observe that $\tau_{f_i,\rho_i} \circ \tau_{f_j,\rho_j} = \tau_{f_i\circ f_j,\rho_i\circ\rho_j}$ and $\delta_{f_i,\rho_i}^\bullet = \delta_{f_i^\bullet,\rho_i^\bullet}$. From this and the fact that $\vee$ commutes with $\bullet$ and $\circ$, we now know $\mathcal{E}$ indeed forms a group, for which the specified semi-unit can be immediately verified to be the identity, using Proposition 2.13 (iv). Moreover if ρ is a symmetry of q then $\vee\{f_i^\bullet \circ \rho_i \circ f_i\} \circ \rho = \vee\{f_i^\bullet \circ \rho_i \circ f_i \circ \rho\} = \vee\{f_i^\bullet \circ \rho_i \circ \rho_i' \circ f_i\}$ where $\rho_i' \in \mathcal{S}(p_i)$ s.t. $\rho_i' \circ f_i = f_i \circ \rho$, using the symmetry preservation of f_i. We then note, for each i, $\rho_i \circ \rho'_i \in \mathcal{S}(p_i)$, and, for each i, j, we have $(\rho_j \circ \rho'_j) \circ \delta_{ij} \circ (\rho_i \circ \rho'_i) = \rho_j \circ \rho'_j \circ f_j \circ (\rho'_i \circ f_i)^\bullet \circ \rho_i = \rho_j \circ f_j \circ \rho \circ \rho^\bullet \circ f_i^\bullet \circ \rho_i = \delta_{ij}$, hence $\vee\{f_i^\bullet \circ \rho_i \circ \rho'_i \circ f_i\}$ is in $\mathcal{E}$ again, similarly for multiplication by a symmetry from the left. Thus, $\mathcal{E}$ indeed gives a presymmetry group on q, from which we know it has a representation, say $g : q \to q_0$. Let $g_i \stackrel{\text{def}}{=} f_i \circ g^\bullet$. Since by construction $\mathtt{pre}(f_i) \le \mathtt{pre}(g)$, we know, using Proposition 2.15 (i), $g_i \circ g_i^\bullet = f_i \circ \mathtt{pre}(g) \circ f_i^\bullet = f_i \circ f_i^\bullet \circ \mathbf{1}_{p_i}$ for each i. Since g gives an isomorphism between $\mathcal{S}(q_0)$ and $\mathcal{E}$ via conjugation, we have, given $\rho_0 \in \mathcal{S}(q_0)$, that $g_i \circ \rho_0 = f_i \circ \tau_{f_i, \rho_i} = \rho_i \circ f_i$ for some $\rho_i \in \mathcal{S}(p_i)$, so f_0 is symmetry-preserving. Thus g_i is a map for each i, and, moreover, $g_j \circ g_i^\bullet = f_j \circ \mathtt{pre}(g) \circ f_i^\bullet = \delta_{ij}$, hence $\{g_i\}$ indeed semi-tabulates $\langle \delta_{ij} \rangle$. For universality the following is essential.

Lemma. If $\{\rho_i \in \mathcal{S}(p_i)\}$ satisfies $\delta_{ij} = \rho_j \circ \delta_{ij} \circ \rho_i^\bullet$ for each i, j, we always have $\vee\{g_i^\bullet \circ \rho_i \circ g_i\} \in \mathcal{S}(q)$ for $\{g_i\}$ given above.

Indeed, $\vee\{g_i^\bullet \circ \rho_i \circ g_i\} = g \circ \vee\{f_i^\bullet \circ \rho_i \circ f_i\} \circ g^\bullet = \rho \circ g^\bullet = \rho$ for $\rho \in \mathcal{S}(q)$, as required. Take, then, any semi-tabulation of $\langle \delta_{ij} \rangle$, say $\{f'_i : q' \to p_i\}$. We then take $h \stackrel{\text{def}}{=} \vee\{g_i^\bullet \circ f'_i\}$ which turns out to be the mediating arrow. First we show this join is well-defined. Indeed $g_i^\bullet \circ f'_i \circ f'^\bullet_j \circ g_j = g_i^\bullet \circ \delta_{ij}^\bullet \circ g_j = g_i^\bullet \circ g_i \circ g_j^\bullet \circ g_j$ which is clearly a semi-unit, similarly for the other case. Next we show h is a map. For the symmetry-preservation, suppose $\rho' \in \mathcal{S}(q')$. We construct $\rho \in \mathcal{S}(q)$ such that $h \circ \rho' = \rho \circ h$ as follows. First, for each i, g_i is a map, hence there is ρ'_i such that $g_i \circ \rho' = \rho'_i \circ g_i$. Now take $\rho \stackrel{\text{def}}{=} \vee\{g_i^\bullet \circ \rho_i \circ f_i\}$. Observing $\delta_{ij} = \rho'_j \circ \delta_{ij} \circ \rho'_i$ holds for each i, j, we know $\rho \in \mathcal{S}(q_0)$ by the Lemma above. Moreover:

$$
\begin{aligned}
\rho' \circ h &= \vee_i\{g_i^\bullet \circ \rho'_i \circ g'_i\} \circ \vee_i\{g_i^\bullet \circ f'_i\} \\
&= \vee_i\{g_i^\bullet \circ \rho'_i \circ f'_i\} \vee (\vee_{i \neq j}\{g_i^\bullet \circ \rho'_i \circ g_i \circ g_j^\bullet \circ f'_j\}) \\
&= \vee_i\{g_i^\bullet \circ \rho'_i \circ f'_i\} \\
&= \vee\{g_i^\bullet \circ f'_i\} \circ \rho \\
&= h \circ \rho,
\end{aligned}
$$

thus h is indeed symmetry-preserving [note the second summand of the join in the second line vanishes because $g_i^\bullet \circ \rho'_i \circ g_i \circ g_j^\bullet \circ f'_j = g_i^\bullet \circ \rho'_i \circ \delta_{ij}^\bullet \circ f'_j = g_i^\bullet \circ \rho'_i \circ f'_i \circ f'^\bullet_j \circ f'_j \le g_i^\bullet \circ \rho'_i \circ f'_i$ by Proposition 2.13 (ii)]. Second we show $h \circ h^\bullet = \mathbf{1}_q$. We first calculate: $(g_i^\bullet \circ f'_i) \circ (g_j^\bullet \circ f'_j)^\bullet = (g_i^\bullet \circ g_i) \circ (g_j^\bullet \circ g_j) = \mathtt{pre}(g_i) \wedge \mathtt{pre}(g_j)$, the last equation using Proposition 2.13 (iii). Note if $i = j$ then the final expression is $\mathtt{pre}(g_i)$ itself. Using this, we can easily calculate $h \circ h^\bullet = \vee\mathtt{pre}(g_i)$. But $\vee\mathtt{pre}(g_i)$ is by definition $g \circ \vee\mathtt{pre}(f'_i) \circ g^\bullet$ but $\vee\mathtt{pre}(f'_i)$ is the identity of $\mathcal{E}$ above, hence this is nothing but $g \circ g^\bullet = \mathbf{1}_q$, as required. Finally we verify uniqueness. Let h' also mediates $\{f'_i\}$ and $\{g_i\}$ as well as h, i.e. we have $f'_i = g_i \circ h'$ for each i. Then $g_i^\bullet \circ f'_i = \mathtt{pre}(g_i) \circ h'$. But $\vee\mathtt{pre}(g_i) = h \circ \vee\mathtt{pre}(f_i) \circ h^\bullet = \mathbf{1}_q$, hence $h' = \vee\{g_i^\bullet \circ f'_i\} = h$, establishing the first statement. The second statement is immediate from the first statement above and Proposition 2.24, Finally "only if" direction is easy by the above construction and noting two tabulations are always mediated by a strong isomorphism in the same way as Proposition 2.9 (i), while for "if" direction we assume such a semi-tabulation exists and construct a tabulation out of it as above. Then we can easily deduce, from the construction of the representation, that it is both-way symmetry preserving and (not only its image but also) its preimage is identity, so that the semi-tabulation itself should be a tabulation, hence done. ■

Some consequences of the above Proposition will also become important later.

2.26 Corollary. (tabulation and equivalence)

(i) *If $\{f_i\}$ semi-tabulates $\langle\delta_{ij}\rangle$ and $f'_i \sim f_i$ for each i, then $\{f'_i\}$ semi-tabulates $\langle\delta'_{ij}\rangle$ such that $\langle\delta'_{ij}\rangle \sim \langle\delta_{ij}\rangle$. Moreover if $\{f_i\}$ tabulates $\langle\delta_{ij}\rangle$ then $\{f'_i\}$ also tabulates $\langle\delta'_{ij}\rangle$. Conversely, if $\{f_i\}$ semi-tabulates $\langle\delta_{ij}\rangle$ and $\langle\delta_{ij}\rangle \sim \langle\delta'_{ij}\rangle$, then there is a semi-tabulation $\{f'_i\}$ of $\langle\delta_{ij}\rangle$ such that, for each i, $f_i \sim f'_i$. Again if $\{f_i\}$ is a tabulation, we can take a tabulation $\{f'_i\}$.*

(ii) *Assume $\{f_i\}$ semi-tabulates $\langle\delta_{ij}\rangle$. If $\{g_i\}$ and $\{g'_i\}$ with the same tabulating object tabulates $\langle\delta_{ij}\rangle$ and, moreover, $g_i \sim g'_i$ for each i, then the corresponding mediating maps from $\{f_i\}$ are equivalent. Conversely, if h mediates a semi-tabulation $\{f_i\}$ and a corresponding tabulation $\{g_i\}$ and $h' \sim h$, then h' is also a mediating map from $\{f_i\}$ to some $\{g'_i\}$ which tabulates the same connection and, moreover, $g_i \sim g'_i$ for each i.*

PROOF:

(i) For the first half of (i), using symmetry preservation, we can put $f'_i = \rho_i \circ f_i$ for each i. Notice $\{f'_i\}$ does semi-tabulate $\{\rho_j \circ \delta_{ij} \circ \rho_i^\bullet\}$, the latter indeed forming a connection. In the case of tabulation, one can easily show (by calculation) that the collection of $\vee f'_i \circ \rho'_i \circ f_i$ with $\{\rho'_i \in \mathcal{S}(p_i)\}$ satisfying $\delta_{ij} = \rho'_j \circ \delta_{ij} \circ \rho'_i$ precisely give symmetries of the (common) domain of tabulations. But by the third statement of Proposition 2.25, this implies $\{f'_i\}$ is a tabulation of $\{\rho_j \circ \delta_{ij} \circ \rho_i^\bullet\}$, as required. For the second half, we just use $f'_i = \rho_i \circ f_i$ assuming $\delta'_{ij} = \rho_j \circ \delta_{ij} \circ \rho_i^\bullet$, which does semi-tabulate $\langle\delta_{ij}\rangle$. The case for tabulation follows from the first half.

(ii) We only show the first half. The second half is easier. By the first statement of Proposition 2.25, the mediating maps are given by $h = \vee\{g_i^\bullet \circ f_i\}$ and $h' = \vee\{g_i'^\bullet \circ f_i\}$. Putting $g'_i = \rho_i \circ g_i$, we get $h' = \vee\{g_i^\bullet \circ \rho_i^\bullet \circ f_i\}$. But by assumption $g_j \circ g_i^\bullet = g'_j \circ g_i'^\bullet = \rho_j \circ g_j \circ g_i^\bullet \circ \rho_i^\bullet$ for each i, j so that $\{\rho_i\}$ gives a symmetry of the connection, whence, by the last statement of the same proposition, there is some $\rho \in \mathcal{S}(q)$ (q being the tabulating object) such that $\rho_i \circ g_i = g_i \circ \rho$. Thus $h' = \vee\{g_i^\bullet \circ \rho_i \circ f_i\} = \rho \circ \vee g_i^\bullet \circ f_i \sim h$, as required. ■

We have now established the cornerstones of theory of connection domains as far as we need in the subsequent development. There are further results concerning them, both in terms of general connection domains and regarding concrete instances of connection domains and their specific properties, as well as variants of the theories. Leaving the exploration of these aspects to elsewhere, we now proceed to define a theory of processes on the basis of what we have built so far.

3 Abstract Process Structure

Proto-processes in a connection domain correspond to *elements* of a set in the setting of the present theory. Connections relate these elements individually. These give the essential tools to develop theory of set-like operations to manipulate processes collectively, which is developed in this section. The first thing to do is to define an analogue of a set itself.

3.1 Definition. (process structure on $\mathcal{C}$) Given a connection domain $\mathcal{C}$, a *process structure on* $\mathcal{C}$ is an indexed family of proto-processes from $\mathcal{C}$. Then $\mathcal{C}$ is the *base domain* of the process structure, and elements of the indexing set are *processes*.

3.2 Notations. $\mathbf{P}, \mathbf{Q}, \mathbf{R}, \ldots$ denote process structures, while $p, q, \ldots$ denote processes. We often treat processes as if they are elements of a process structure, writing, say, $p \in \mathbf{P}$. Then we write $\underline{p}, \underline{q}, \ldots$ to denote proto-processes (i.e. objects in $\mathcal{C}$) which underlie $p, q, \ldots$.

3.3 Examples.

(i) A process structure on the one-arrow, one-object connection domain Elm of Example 2.19 is an indexed family of the unique object of Elm, hence is essentially its indexing set itself.

(ii) A process structure on $\mathsf{FinPerm}$ is precisely a process structure in the sense of our preceding Notes. As proved in [7], this structure has an equivalent presentation which directly underlies many mathematical and computational structures studied so far. In this form, the structure relates to a series of study of Aczel [2, 3] and Nerode [13] where algebra of open terms involving general substitution is considered. We shall discuss their theory (especially that of [3]) in comparison with our algebraic theory over process structures in Part II.

(iii) A process structure on Perm generalises the above to the case when there can be infinite connection points in each process. This structure is useful for some applications.

(iv) If we introduce webs to Perm or $\mathsf{FinPerm}$ as discussed in Example 2.19, we get theory of webs for processes. Both in terms of basic mathematics of process structures as well as for applications, this theory is essential.

(v) Similarly a process structure on Top consists of processes each of which is a topological space and a subgroup of auto-homeomorphisms. One can think of adding webs to Top, which can again be handled by identification topology.

We now define what corresponds to a relation over sets.

3.4 Definition.

(i) (correspondence) The triple of the form $\langle p,\ \delta,\ q\rangle$ with p and q being processes from process structures with the same base and δ being a connection from $\underline{p}$ and $\underline{q}$, is called a *correspondence from p to q*, where δ is its *component connection.* We often write $p\Re_\delta q$ for $\langle p,\ \delta,\ q\rangle \in \Re$.

(ii) (p-relation) Let $\mathbf{P}$ and $\mathbf{Q}$ be two process structures with the same base domain. Then a *p-relation* from $\mathbf{P}$ to $\mathbf{Q}$ is a set $\Re$ of correspondences such that: (1) if $\langle p,\ \delta,\ q\rangle \in \Re$ then $p \in \mathbf{P}$ and $q \in \mathbf{Q}$ and (2) $\Re$ be closed under $\sim$, i.e. $\langle p,\ \delta,\ q\rangle \in \Re$ and $\delta \sim \delta'$ imply $\langle p,\ \delta',\ q\rangle \in \Re$. Then we say $\mathbf{P}$ is a *domain* and $\mathbf{Q}$ is a *codomain* of $\Re$.

(iii) (operations on p-relation) If $\Re$ and $\Re'$ are p-relations from $\mathbf{P}$ to $\mathbf{Q}$, such operations as $\Re \cup \Re'$, $\Re \cap \Re'$ and $\Re \backslash \Re'$ are naturally defined, where the first two naturally extends to a family of p-relations. The set of p-relations from $\mathbf{P}$ to $\mathbf{Q}$ form, under these operations, a complete boolean lattice, as is easily verified. We also define:

$$\Re^{\bullet} \stackrel{\text{def}}{=} \{\langle q,\ \delta^{\bullet},\ p\rangle \mid \langle p,\ \delta,\ q\rangle \in \Re\}$$

which is the *inverse* of $\Re$, and is indeed a p-relation again (with the domain and codomain exchanged). P-relations also compose:

$$\Re_1\,\Re_2 \stackrel{\text{def}}{=} \{\langle p,\ \delta_2 \circ \delta_1,\ r\rangle \mid \exists q.\ \ (\langle p,\ \delta_1,\ q\rangle \in \Re_1\ \wedge\ \langle q,\ \delta_2,\ r\rangle \in \Re_2)\}$$

where we assume the domain of $\Re_2$ coincides with the codomain of $\Re_1$. This is again a p-relation, by Proposition 2.3.

Then we easily obtain:

3.5 Proposition. *The class of all process structures with base $\mathcal{C}$ and p-relations between them form a category where the identity on $\mathbf{P}$ is given by $\{\langle p,\ \rho,\ p\rangle \mid \rho \in \mathcal{S}(\underline{p}), p \in \mathbf{P}\}$, written $\mathbf{ID_P}$. The category is denoted $\mathsf{PS}^{\mathcal{C}}_{rel}$.*

As in [7], we may equip each homset of such a category with relational operations of Definition 3.4. Some examples:

3.6 Examples.

(i) $\mathsf{PS}_{rel}^{\mathsf{Elm}}$ is essentially the familiar category of relations. Note, if $\langle e_1,\ \delta,\ e_2\rangle$ is a correspondence over process structures on Elm, then δ is uniquely determined as the unique arrow in Elm, i.e. a correspondence here is essentially a *tuple* of e_1 and e_2.

(ii) $\mathsf{PS}_{rel}^{\mathsf{FinPerm}}$ is precisely what we have known as PS_{rel} in the preceding Notes. We have shown in [7] that this category is equivalent to the category of compatible relations over so-called *rooted process structures*, which often underlies the foregoing study of semantics of processes, or more generally that of computation.

(iii) If we take Top as a base category, a correspondence relates two processes by their "parts" injectively. We can understand this as a kind of identification, and, as noted, the resulting topological process is induced by the standard identification topology. A p-relation in this setting collects such correspondences: the identity p-relation consists of all homeomorphisms on each topological process as correspondences.

We next introduce a special kind of p-relations, which behaves quite analogously to functions in the present setting.

3.7 Definition. (p-map) A *p-map* is a p-relation in which there is one and only one correspondence from each process of the domain modulo $\sim$, and whose component connections are all maps (in the base category). A *p-isomorphism* is a p-map whose inverse is also a p-map.

We often write $F, G, \ldots$ to denote p-maps. The composition of two p-maps (as p-relations) is written $G \circ F$ (which is p-relationally $F\,G$). By Proposition 2.9 (iii), $G \circ F$ is again a p-map. We often write $F : p \mapsto_f q$ for $\langle p,\ f,\ q\rangle \in F$ (which is p-relationally $pF_f q$). We also note:

3.8 Proposition. *Given a p-relation $\Re : \mathbf{P} \to \mathbf{Q}$, both $\Re\,\Re^\bullet = \mathbf{ID}_\mathbf{P}$ and $\Re^\bullet\,\Re = \mathbf{ID}_\mathbf{Q}$ hold if and only if $\Re$ is a p-isomorphism.*

PROOF: "If" direction is easy. For "only if" direction, clearly $\Re$ relates processes bijectively (if not $\Re\,\Re^\bullet$ and $\Re^\bullet\,\Re$ cannot be identity). Thus if elements of $\Re$ are maps, we are done. Assume $\langle p,\ \delta,\ q\rangle \in \Re$. We first show δ and $\delta^\bullet$ are mutually inverse to each other. By axiom on connection, $\delta \circ \delta^\bullet \circ \delta = \delta$. Note, by assumption, for some $\delta' \sim \delta$ we have $\delta \circ \delta'^\bullet = \rho \in \mathcal{S}(q)$. But this shows $\delta \circ (\rho \circ \delta'^\bullet) = \mathbf{1}_q$. Multiplying this right inverse to the axiom above, we know $\delta^\bullet$ indeed is the right inverse of δ, hence done. Similarly for $\delta^\bullet \circ \delta = \mathbf{1}$. Now we show δ is symmetry-preserving. Let $\rho \in \mathcal{S}(p)$ and $\rho' \stackrel{\text{def}}{=} \delta \circ \rho \circ \delta^\bullet$. Then by assumption $\langle p,\ \rho',\ p\rangle \in \mathbf{ID}$, i.e. $\rho' \in sym(q)$. But

$$\delta \circ \rho \;=\; (\delta \circ \rho) \circ (\delta^\bullet \circ \delta) \;=\; \rho' \circ \delta,$$

as required. Thus δ is a map, indeed a strong isomorphism, hence done. ■

Immediately:

3.9 Corollary. *The collection of process structures with base $\mathcal{C}$ and p-maps between them forms a subcategory of $\mathsf{PS}^{\mathcal{C}}_{rel}$, which we denote $\mathsf{PS}^{\mathcal{C}}$. The isomorphisms in $\mathsf{PS}^{\mathcal{C}}$ are precisely p-isomorphisms, which coincide with the isomorphisms in $\mathsf{PS}^{\mathcal{C}}_{rel}$.*

3.10 Example.

(i) $\mathsf{PS}^{\mathsf{Elm}}$ is (essentially) the category of sets. Indeed we know, by Definition 3.7 and comments in (i) of Example 3.6, that a p-map between two sets is precisely a function in the usual sense.

(ii) $\mathsf{PS}^{\mathsf{FinPerm}}$ is what we have known as PS in the preceding study of process structures, cf. [7].

(iii) We can similarly consider $\mathsf{PS}^{\mathsf{Top}}$ as an example. Note here a map maps a "part of" a topological process to the "whole of" another topological process, preserving the original symmetry (and perhaps adding some more). Thus a p-map gives, for each topological process in the source, its partial image as a topological process in the target.

We already know that a correspondence (or, more exactly, a connection), is itself a proto-process. Using this we can regard a p-relation as a process structure, just as a relation over sets again forms a set.

3.11 Definition. Given a p-relation $\Re$, we write $\underline{\Re}$, or just $\Re$ if no confusion arises, to denote a process structure whose processes are given by:

$$\{\langle p,\ \delta,\ q\rangle \mid \langle p,\ \delta,\ q\rangle \in \Re\} \,/\, \sim, \tag{3}$$

where, in each equivalence class, the proto-process underlying it is specified as the tabulation of a connection of any selected correspondence in it. Note, by Proposition 2.26 (ii), any such selection results in strongly isomorphic proto-processes, hence p-isomorphic process structures.

Using the notation introduced in Section 2, an element of a p-relation as a process structure would be written $\langle p,\ \underline{\delta},\ q\rangle$. Based on the construction, the following states that $\mathsf{PS}^{\mathcal{C}}$ has finite products, regardless of a base domain $\mathcal{C}$. The proof will be given with greater generality soon, hence is omitted.

3.12 Definition and Proposition. (finite products in $\mathsf{PS}^{\mathcal{C}}$)

(i) *The process structure consisting of only one process whose underlying proto-process is the terminator of the base domain is the terminator in $\mathsf{PS}^{\mathcal{C}}$.*

(ii) *Let us denote the p-relation from* $\mathbf{P}$ *to* $\mathbf{Q}$ *which consists of all correspondences from processes in the former to those in the latter by* $\mathbf{P} \times \mathbf{Q}$*, called the* product *of* $\mathbf{P}$ *and* $\mathbf{Q}$*. Then* $\mathbf{P} \times \mathbf{Q}$ *as a process structure is a categorical product of* $\mathbf{P}$ *and* $\mathbf{Q}$ *in* $\mathsf{PS}^{\mathcal{C}}$*.*

For our later purpose we need to deal with a product over an arbitrary (small) family of process structures. As expected, this is where "generalised connections" introduced in Definition 2.22 comes in. Below we fix a base domain $\mathcal{C}$ as always.

3.13 Definition.

(i) (generalised correspondence) Given a family of process structure on $\mathcal{C}$, say $\{\mathbf{P}_i\}_{i\in I}$, a *correspondence in* $\{\mathbf{P}\}$ is a pair of $\{p_i \in \mathbf{P}_i\}$ and $\{\delta_{ij} : \underline{p_i} \to \underline{p_j}\}$ such that the latter forms a connection over $\{\underline{p_i}\}$ in the base domain $\mathcal{C}$ and, moreover, it has a semi-tabulation in $\mathcal{C}$ (we then speak of a correspondence *over* $\{p_i\}$). Such a correspondence is often denoted $\langle \delta_{ij} : \{p_i\}\rangle$ or $\langle \delta_{ij}\rangle$ for short. When the family is finite, we also write $\langle \delta_{ij} : p_1, .., p_n\rangle$. Then we define:

$$\langle \delta_{ij} : \{p_i\}\rangle \ \sim\ \langle \delta'_{ij} : \{p_i\}\rangle \ \stackrel{\text{def}}{\Leftrightarrow}\ \langle \delta_{ij}\rangle \ \sim\ \langle \delta'_{ij}\rangle \ \text{ in } \mathcal{C}.$$

(note $\{p_i\}$ should coincide between two correspondences).

(ii) (process product of a family of process structures) Given a family of process structures $\{\mathbf{P}_i\}$, we define the *process product over* $\{\mathbf{P}_i\}$, denoted $\prod \mathbf{P}_i$, as a set of all correspondences in $\{\mathbf{P}_i\}$.

Observe that we require the underlying connection of a correspondence to have a semi-tabulation (hence a tabulation, by Proposition 2.25). This is essential to make it the categorical product in $\mathsf{PS}^{\mathcal{C}}$. Also note that a generalised correspondence becomes the correspondence in the sense of Definition 3.4 (i) if it is over a pair of processes. Similarly the only correspondence in the empty family of process structures is a pair of the empty set and the 0-ary connection in the base domain.

We now prove the generalisation of Proposition 3.12. This becomes essential for our main result in Part Two.

3.14 Proposition. (existence of all small products in $\mathsf{PS}^{\mathcal{C}}$) *Assume given a product* $\prod \mathbf{P}_i$ *of* $\{\mathbf{P}_i\}$*. Then a process structure* $\underline{\prod \mathbf{P}_i}$*, or often just written* $\prod \mathbf{P}_i$ *if no confusion is possible, is given by equivalence classes* $\{[\langle \delta_{ij}\rangle]_\sim \mid \langle \delta_{ij}\rangle \in \prod \mathbf{P}_i\}$ *as processes whose underlying proto-processes are, for each equivalence class, the tabulating object of the generalised connection of its selected element (which exists since it has a semi-tabulation by definition). Then* $\prod \mathbf{P}$ *gives a product of* $\{\mathbf{P}_i\}$ *in* $\mathsf{PS}^{\mathcal{C}}$*.*

Proof: Below $\prod \mathbf{P}_i$ denotes a product as a process structure as given above. First we construct projection arrows. For each i, define $\Pi_i : \prod \mathbf{P}_i \to \mathbf{P}_i$ as follows. Assuming a selected tabulation for each tabulating object underlying a process in $\prod \mathbf{P}_i$, we set:

$$\Pi_i \stackrel{\text{def}}{=} \{\langle[\langle\delta_{ij}\rangle]_\sim,\ \pi'_i,\ p_i\rangle \mid \text{For a selected tabulation } \{\pi_i\} \text{ of } \langle\delta_{ij}\rangle \text{ we have } \pi'_i \sim \pi_i\}$$

where we write $[\langle\delta_{ij}\rangle]_\sim$ for $[\langle\delta_{ij} : \{p_i\}\rangle]_\sim$ since no confusion is possible. We first show Π_i is a p-map. Indeed, by definition it is defined everywhere in the domain, its constituents are all maps, and it is closed under symmetries. We now establish universality. Suppose there exist a family of p-maps $\{F_i : \mathbf{R} \to \mathbf{P}_i\}$. Take any $r \in \mathbf{R}$ and $\{f_i : r \to p_i\}$ such that $F : r \mapsto_{f_i} p_i$ (note $\{p_i\}$ is determined uniquely by r since each F_i is a p-map, but $\{f_i\}$ is determined only up to individual equivalences). Then $\{f_i\}$ semi-tabulates $\langle\delta'_{ij}\rangle$ given by $\delta'_{ij} = f_j \circ f_i^\bullet$, cf. discussions after Definition 2.22. Now $\langle\delta'_{ij} : p_i\rangle$ is a correspondence with a semi-tabulation, hence $[\langle\delta'_{ij} : p_i\rangle]_\sim$ is in $\prod \mathbf{P}_i$. Now we claim the tabulation of $\langle\delta'_{ij}\rangle$ is in $\{\Pi_i\}$. Indeed at least there is a tabulation $\{\pi_i \in \Pi_i\}$ of $\langle\delta_{ij}\rangle \sim \langle\delta'_{ij}\rangle$, but, by the second half of Corollary 2.26 (ii) and our assumption, it follows that there is, say, $\{\pi'_i \in \Pi_i\}$ which tabulate $\langle\delta'_{ij}\rangle$, as required. But this means, due to the universality of a tabulation, that there is a unique map h from r to $[\langle\delta'_{ij} : p_i\rangle]_\sim$ such that $f_1 = \pi'_1 \circ h$. This gives individual constructs for the required mediating p-map, and indeed we take, for each r:

$$H_r \stackrel{\text{def}}{=} \{\langle r,\ h,\ [\langle\delta_{ij} : p_i\rangle]_\sim\rangle \mid \langle\delta_{ij} = f_j \circ f_i^\bullet\rangle,\ F : r \mapsto_{f_i} p_i, h \text{ given as above.}\}$$

Then we define:

$$H \stackrel{\text{def}}{=} \bigcup\nolimits_{r\in\mathbf{R}} H_r.$$

We now show H is a p-map. First, by Corollary 2.26 (i), we know that if $\langle r,\ h,\ [\langle\delta_{ij} : p_i\rangle]_\sim\rangle \in H_r$ and $\langle r,\ h',\ [\langle\delta'_{ij} : p_i\rangle]_\sim\rangle \in H_r$, then $[\langle\delta_{ij} : p_i\rangle]_\sim = [\langle\delta'_{ij} : p_i\rangle]_\sim$. Moreover by the clause (ii) of the same corollary, we know $h \sim h'$, whence it is unique up to $\sim$. Finally by the second half of the same clause, we know H_r is closed under symmetries, as required. Now by construction it is always the case that if $H : r \mapsto_h [\langle\delta_{ij}\rangle]_\sim$ and $\Pi_i : [\langle\delta_{ij}\rangle]_\sim \mapsto_{\pi_i} p_i$, then for some $\langle r,\ f_i,\ p_i\rangle \in F_i$ we have $f_i = \pi_i \circ h$, from which we know $\Pi_i \circ H = F_i$ for each i. Finally we should check the uniqueness of H. Suppose, in addition to the above H_r, there is another p-relation H'_r from $\mathbf{R}$ to $\mathbf{P} \times \mathbf{Q}$ whose elements are all from r with component connections being maps, and, moreover, which satisfies, for each i,

$$\text{(4)} \qquad \Pi_i \circ H_r \;=\; \Pi_i \circ H'_r \;=\; \{\langle r,\ f,\ p_i\rangle \in\ F_i\}.$$

Now by Corollary 2.26 (i), for any $\langle\delta_{ij}\rangle$, $\langle\underline{\delta'_{ij}},\ \pi'_i,\ p_i\rangle \in \Pi_i$, π'_i and π'_2 always tabulate some $\langle\delta''_{ij}\rangle \sim \langle\delta'_{ij}\rangle$. Set $f'_i \stackrel{\text{def}}{=} \pi'_i \circ h'$. Then if in addition $\langle r,\ h',\ [\delta'_{ij}]_\sim\rangle \in H'_r$, it should be the case that h' is the unique mediating arrow from the semi-tabulation $\{f'_i\}$ to the tabulation $\{\pi'_i\}$. But by (4), we know $f'_j \circ f'^\bullet_i \sim f_j \circ f_i{}^\bullet$ with $\langle r,\ f_i,\ p_i\rangle \in F_i$, which implies, with $\langle r,\ h,\ [\delta_{ij}]_\sim\rangle \in H_r$, we have $h' \sim h$ and $\underline{\langle\delta_{ij}\rangle} = \underline{\langle\delta'_{ij}\rangle}$, as required. ■

We also note, without proof, that:

3.15 Proposition. $\mathsf{PS}^{\mathcal{C}}$ *is cartesian (hence is complete) with exponentials.*

This holds taking any base domain $\mathcal{C}$. Existence of exponentials, which is the only non-trivial part of the verification and uses the existence of representations in essential ways, becomes important in some algebraic constructions, but we shall not use them here hence we omit the concrete construction. Now once we know the above result, it is easy to deduce:

3.16 Proposition. $\mathsf{PS}^{\mathcal{C}}$ *is a topos.*

(The subobject classifier is given by a process structure with two processes, the underlying proto-process for both being the terminator of the base domain.) The above fact is useful for several proofs and development, since it makes many standard constructions and results available to us (conspicuous examples will be found in Part Two of the present Notes). At the same time, it is notable that it is *not* the case every "standard" construction gives us precisely what we want. One symptomatic fact is that the "relational universe" which can be constructed from the topos $\mathsf{PS}^{\mathcal{C}}$ in the standard way [4, 10] does *not* give $\mathsf{PS}^{\mathcal{C}}_{rel}$ in general (even though all p-relations can be presented as monic pairs). Concretely such mismatch results in, for example, that equivalence relations defined categorically only give a subclass of equivalences we shall define later, though standard construction of kernel pairs is not uninteresting as we shall comment later. We do not detail these points further but merely point out that this is an inevitable consequence of treating objects more complex than elements as the basic constituents, and of assuming that some structure of each object can be forgotten in relational construction. This has been implicit in the preceding treatment of e.g. behavioural equivalences over processes in process algebras, or, as Peter Aczel pointed out, even in such basic mathematical notions as polynomials over a field. Thus categorical properties can be indeed exploited usefully (as we shall often do later), one should take care on what is exactly represented by given categorical notions if ever we are using them. Our basic attitude would be, then, if some result can be gained in a straightforward manner using concrete constructions, let us use them, while categorical counterpart may be mentioned to give further insight on the matter: when the abstract framework is really needed to travel further, however, we shall rely on them without hesitation.

Here we proceed to another concrete construction, the quotient of a process structure. The direct construction of quotient from "equivalence" in the present setting reveals essential role the notion of representation plays in our process theory. Notice we fix the base domain, as we always do.

3.17 Definition. Suppose $\mathcal{E} \subset hom(p,p)$ is such that $\mathcal{S}(p) \subset \mathcal{E}$ and it is closed under $\circ$ and $(\ )^{\bullet}$. Then $\mathcal{E}$-*effective semi-unit*, written $p|\mathcal{E}$, is given by $\wedge\{$semi-units in $\mathcal{E}\}$.

By Proposition 2.20, $p|\mathcal{E}$ is itself a semi-unit. We also note:

3.18 Proposition and Definition. *Assume given $\mathcal{E} \in hom(p,p)$ which contains $\mathcal{S}(p)$ and is closed under $\circ$ and $(\ )^\bullet$ as above.*

(i) *Suppose $p|\mathcal{E} \in \mathcal{E}$. Then $(p|\mathcal{E}) \circ \mathcal{E} \circ (p|\mathcal{E})$ gives the maximum presymmetry group in $\mathcal{E}$ whose identity is $p|\mathcal{E}$, which also coincides with the set of minimal elements of $\mathcal{E}$. We call this presymmetry group the* center *of $\mathcal{E}$.*

(ii) *Suppose $p|\mathcal{E} \notin \mathcal{E}$. Then the* extension *of $\mathcal{E}$ is the set, say $\mathcal{E}'$, given by closing $\mathcal{E} \cup \{p|\mathcal{E}\}$ under $\circ$ (and hence $(\ )^\bullet$). Then we have $p|\mathcal{E}' = p|\mathcal{E}$.*

PROOF:

(i) Set $\mathcal{E}_0 \stackrel{\text{def}}{=} (p|\mathcal{E}) \circ \mathcal{E} \circ (p|\mathcal{E})$. Then it is easy to see that $\mathcal{E}_0$ forms a pre-symmetry group whose identity is $p|\mathcal{E}$. Suppose there is a pre-symmetry group within $\mathcal{E}$ whose identity is $p|\mathcal{E}$ and δ is its element. Then $\delta \circ p|\mathcal{E} = p|\mathcal{E} \circ \delta = \delta$ which shows $p|\mathcal{E} \circ \delta \circ p|\mathcal{E} = \delta$ by which we know $\delta \in \mathcal{E}_0$, as required. We next show if δ is minimal in $\mathcal{E}$ then it is in $\mathcal{E}_0$. Assume δ is minimal in $\mathcal{E}$. Then $p|\mathcal{E} \circ \delta \leq \delta$ [which is from Proposition 2.13 (ii)] shows $p|\mathcal{E} \circ \delta = \delta$. Similarly $\delta \circ p|\mathcal{E} \circ \delta = \delta$, by which $p|\mathcal{E} \circ \delta \circ p|\mathcal{E} = \delta$, i.e. $\delta \in \mathcal{E}_0$. But if $\delta_0 \in \mathcal{E}_0$ and $\delta(\in \mathcal{E}) \leq \delta_0$, then $\delta^\bullet \circ \delta \leq \delta_0^\bullet \circ \delta_0 = p|\mathcal{E}$ so that $\delta^\bullet \circ \delta = p|\mathcal{E}$. Multiplying both sides by δ_0, we get $\delta = \delta_0$ by (2.11), hence done.

(ii) Since any element in $\mathcal{E}'$ cannot go below $p|\mathcal{E}$. ∎

The above result motivates the following notation.

3.19 Definition. If $\mathcal{E} \in hom(p,p)$ contains $\mathcal{S}(p)$ and is closed under $\circ$ and $(\ )^\bullet$, we write $\langle\mathcal{E}\rangle$ for the center of $\mathcal{E}$ if it contains $p|\mathcal{E}$, or else for the center of the extension of $\mathcal{E}$ as defined in 3.18 (ii).

We are now ready to introduce one essential construction in our theory, the *quotient process structure*. Let us say a p-relation on $\mathbf{P}$ (i.e. from $\mathbf{P}$ to $\mathbf{P}$) is *reflexive* when $\mathbf{ID_P} \subset \Re$, *symmetric* when $\Re^\bullet = \Re$, and *transitive* when $\Re\Re \subset \Re$. It goes without saying that:

3.20 Definition. A reflexive, symmetric, and transitive p-relation is called an *equivalence p-relation.*

To construct a quotient from an equivalence, we use existence of a representation of the presymmetry in an essential way. First an equivalence induces, on each process it relates, a set of endo-connections in the following way.

3.21 Definition. Let $p \in \mathbf{P}$ and $\Re$ be an equivalence on $\mathbf{P}$. The *restriction of* $\Re$ *on* $p \in \mathbf{P}$ is a sub p-relation of $\Re$ whose domain and codomain is restricted to $\{p\}$. We write $\underline{\Re|p}$ for the underlying endo-connections on $\underline{p}$.

Note that, in the above, $\underline{\Re|p}$ is closed under $\circ$ and $(\ )^\bullet$ because $\Re$ is an equivalence, hence, by Proposition 3.18, either itself or its extension has a center, written (in both cases) $\langle\underline{\Re|p}\rangle$. We then get the following.

3.22 Proposition. *Given* $\mathbf{P}$ *and an equivalence* $\Re$ *on* p*, suppose* $p\Re_\delta q$ *for some* δ*. Then the following holds.*

(i) $\underline{\Re|p}$ *has a center if and only if* $\underline{\Re|q}$ *has a center.*
(ii) *The representing object of* $\langle\underline{\Re|p}\rangle$ *and that of* $\langle\underline{\Re|q}\rangle$ *are strongly isomorphic.*

To prove, we need a lemma:

3.23 Lemma. *If* u *is a semiunit on* p *and* $\delta : p \to q$ *for some* q*, then* $\delta \circ u \circ \delta^\bullet$ *is a semi-unit on* q*.*

Proof: First $\delta \circ u \circ \delta^\bullet \circ \delta \circ u \circ \delta^\bullet = \delta \circ \delta^\bullet \circ \delta \circ u \circ u \circ \delta^\bullet = \delta \circ u \circ \delta^\bullet$. Second obviously $(\delta \circ u \circ \delta^\bullet)^\bullet = \delta \circ u \circ \delta^\bullet$, hence done. ■

In the proof below, we write the identity of the center $\underline{\Re|p}$ as $\underline{p}|\Re$, which, to be precise, should have been written $\underline{p}|(\underline{\Re|p})$.

Proof (of Proposition 3.22)

(i) Suppose $\underline{p}|\Re$ is in $\underline{\Re|p}$. Write it, for simplicity, e_p. Define $e \stackrel{\text{def}}{=} \delta \circ p|\Re \circ \delta^\bullet$. Note, by transitivity of $\Re$, we immediately know $e \in \Re|q$. We now show $e = q|\Re$. First e is a semi-unit since $p|\Re$ is, by Lemma 3.23. Next suppose $u \in \Re|q$ is a semi-unit. Then we set:

$$\begin{aligned} u \circ e &= (u \circ e \circ e\ =)e \circ u \circ e \\ &= \delta \circ \underline{p}|\Re \circ \delta^\bullet \circ u \circ \delta \circ \underline{p}|\Re \circ \delta^\bullet \\ &= \delta \circ \underline{p}|\Re \circ u' \circ \underline{p}|\Re \circ \delta^\bullet \end{aligned}$$

where $u' = \delta^\bullet \circ u \circ \delta$ is a semi-unit on $\underline{p}$, again from Lemma 3.23. But then $\underline{p}|\Re \leq u'$, hence

$$u \circ e \ \geq\ \delta \circ \underline{p}|\Re \circ \underline{p}|\Re \circ \underline{p}|\Re \circ \delta^\bullet \ =\ \delta \circ \underline{p}|\Re \circ \underline{p} \circ p|\Re \circ \delta^\bullet \ =\ e$$

which shows $e \leq u$, hence done.

(ii) Let $\delta' = \underline{q}|\Re \circ \delta \circ \underline{p}|\Re$ and i, j be representations of the centers of p and q, respectively. Note $\delta'^{\bullet} \circ \delta = \underline{p}|\Re$ and $\delta \circ \delta'^{\bullet} = \underline{q}|\Re$. We show both $j \circ \delta' \circ i^{\bullet}$ and $i \circ \delta'^{\bullet} \circ j^{\bullet}$ are p-maps, by which the statement is proved. Since the argument is symmetric, we only show the case of $j \circ \delta' \circ i^{\bullet}$. Let p_0 and q_0 be the codomain of i and j, respectively, and let $\rho \in \mathcal{S}(p_0)$. Then, using the fact that representation induces isomorphism between the presymmetry group of the original proto-process and the symmetries of the target proto-process, we have, assuming $\rho' \in \langle \Re | p \rangle$,

$$\begin{aligned} j \circ \delta' \circ i^{\bullet} \circ \rho &= j \circ \delta' \circ \rho' \circ i^{\bullet} \\ &= j \circ \delta' \circ \rho' \circ \underline{p}|\Re \circ i^{\bullet} \\ &= j \circ (\delta' \circ \rho' \circ \delta'^{\bullet}) \circ \delta' \circ i^{\bullet}. \end{aligned}$$

But $\delta' \circ \rho' \circ \delta'^{\bullet} \in \langle Re|q \rangle$, hence the final formula becomes, for some $\rho'' \in \mathcal{S}(q)$, $\rho'' \circ \delta' \circ i^{\bullet}$, which shows $j \circ \delta' \circ i^{\bullet}$ is symmetry preserving. But we note:

$$\begin{aligned} (j \circ \delta' \circ i^{\bullet}) \circ (j \circ \delta' \circ i^{\bullet})^{\bullet} &= j \circ \delta' \circ i^{\bullet} \circ i \circ \delta'^{\bullet} \circ j^{\bullet} \\ &= j \circ \underline{p}|\Re \circ j^{\bullet} \\ &= j \circ j^{\bullet} \circ j \circ j^{\bullet} \\ &= \mathbf{1}_q, \end{aligned}$$

hence done. ■

3.24 Definition. (equivalence classes and quotient) Let $\Re$ be an equivalence on $\mathbf{P}$.

(i) The *equivalence class* of $p \in \mathbf{P}$ by $\Re$, written $[p]_{\Re}$, is given by:

$$[p]_{\Re} \stackrel{\text{def}}{=} \{q | p \Re_{\delta} q \text{ for some } \delta\}$$

(ii) A *quotient* of $\mathbf{P}$ by $\Re$ is a process structure whose processes are equivalence classes by $\Re$ and, for each equivalence class $[p]_{\Re}$, the underlying proto-process is given by the representation of $\langle \Re | p' \rangle$ for a selected process $p' \in [p]_{\Re}$.

By Proposition 3.22 we immediately know:

3.25 Proposition. *Two quotients of an equivalence $\Re$ are p-isomorphic.*

Thus we write $\mathbf{P}/\Re$ for a selected quotient of $\mathbf{P}$ by $\Re$, without any loss of generality.[1]

[1] Note we depend on Axiom of Choice in this way. We can avoid this by using "group presentation" as developed for a special case of FinPerm in [8]. We prefer the present approach since the ideas can be presented more intuitively in this way. Mathematically they just give the equivalent theory.

3.26 Remark. Unlike in the case of a set, several equivalence p-relations would induce a single quotient in the present setting. This is because, when $p \sim_\delta q$ and $\delta \leq \delta'$ hold, adding $p \sim_{\delta'} q$ does not change the situation since, in that case, neither $\langle \Re | p \rangle$ nor $\langle \Re | q \rangle$ does not change (this has already been discussed in [8] in the setting of $\mathsf{FinPerm}$). Thus one may define the following canonical ways of presenting equivalences.

(i) A "gross" equivalence: this is an equivalence p-relation $\sim$ such that, whenever $p \sim_\delta q$ and $\delta \leq \delta'$ we have $p \sim_{\delta'} q$.

(ii) A "lean" equivalence: this is an equivalence p-relation $\sim$ such that $\sim = \mathbf{ID} \cup \Re$ where if $\Re$ consists only of minimal correspondences, i.e. $p\Re_\delta q$ and $p\Re_{\delta'}q$ then $\delta \asymp \delta'$ implies $\delta = \delta'$.

(iii) A "lean" pre-equivalence: this is a transitive and symmetric p-relation such that (1) $p\Re_\delta p$ implies $p\Re_{\rho_2 \circ \delta \circ \rho_2} p$ for any $\rho_1, \rho_2 \in \mathcal{S}(p)$, and: (2) if $p\Re_\delta q$ and $p\Re_{\delta'}q$ then $\delta \asymp \delta'$ implies $\delta = \delta'$.

Thus, in a gross equivalence, we have all possible correspondences to induce a quotient, while a lean pre-equivalence owns minimal correspondences to induce a quotient (indeed we can take $\{\delta \mid p\Re_\delta p\}$ itself for a presymmetry group). It is notable that, for any p-map F, $F^{-1} \circ F$ gives a lean pre-equivalence. On the other hand, we know the following.

Fact. In $\mathsf{PS}^{\mathcal{C}}$, kernel pairs give gross equivalences. Conversely, any gross equivalence can be presented by a kernel pair.

The circumstances around the above fact will be discussed elsewhere. We also note that the above fact can be used to give another proof of Proposition 3.27 below.

The final result of this section gives the expected coherence of the theory of homomorphisms and quotient. It becomes a cornerstone of theory of algebra in the subsequent sections.

3.27 Proposition.

(i) *Let $F : \mathbf{P} \to \mathbf{Q}$ be a p-map. Define $\sim_F$ as $\{\langle p,\ \delta,\ q\rangle \mid \exists \delta_0.\ \langle p,\ \delta_0,\ q\rangle \in F^{-1} \circ F$ and $\delta \geq \delta_0\}$.*[2] *Then $\sim_F$ is an equivalence on $\mathbf{P}$.*

(ii) *Let $\sim$ be an equivalence on $\mathbf{P}$ and $\mathbf{P}/\sim$ be its quotient. Define $\eta_\sim : \mathbf{P} \to \mathbf{P}/\sim$,* the natural p-map of $\sim$, *as:*

$$F : p \mapsto_i [p]_\sim \ \text{ iff } i \text{ is the representation of } \langle \Re | p \rangle.$$

then $\eta_\sim$ is a p-map.

[2] Note $\sim_F$ is a "gross equivalence" discussed in Remark 3.26 above: we can take any equivalences which induce the same quotient to gain the same result.

(iii) *If $F : \mathbf{P} \to \mathbf{Q}$ is a surjective p-map and $\sim_F$ is the equivalence defined as in* (i) *above, then there is a unique p-isomorphism I from $\mathbf{P}/{\sim_F}$ such that $F = I \circ \eta_{\sim_F}$ (see the following diagram):*

$$\begin{tikzcd} \mathbf{P} \arrow[r, "\eta_{\sim_F}"] \arrow[d, "F"'] & \mathbf{P}/\sim_F \arrow[dl, dashed, "I"] \\ \mathbf{Q} & \end{tikzcd}$$

Before proving Proposition 3.27, we lists a simple but useful lemma, which holds in an arbitrary connection predomain.

3.28 Lemma. *If f is a map, then $f \sim f'$ implies* $\mathtt{pre}(f) = \mathtt{pre}(f')$.

Proof: If $f \sim f'$, by symmetry preservation, we can write $f' = \rho \circ f$ where ρ is a symmetry of the target. Then $f'^{\bullet} \circ f' = f^{\bullet} \circ \rho^{\bullet} \circ \rho \circ f = f^{\bullet} \circ f$, as required. ∎

Proof (of Proposition 3.27)

(i) We show $\sim_F$ is reflexive, symmetric, and transitive. For reflexivity one should notice that if $\rho \in \mathcal{S}(p)$ and $F : p \mapsto_f q$ then it is always the case that $\rho \geq \rho \circ f^{\bullet} \circ f$, hence $\rho \in \sim_F$, as required. For symmetry and transitivity, we use a p-relation $\sim \stackrel{\mathrm{def}}{=} F^{-1} \circ F$ (this is the "lean pre-equivalence" mentioned in Remark 3.26). For symmetry, notice $\sim$ is obviously symmetric. But suppose $p {\sim_F}_{\delta} q$. Then $p \sim_{\delta_0} q$ with $\delta_0 \leq \delta$. By symmetry of δ_0, we have $q \sim_{\delta_0^{\bullet}} p$. But $\delta_0^{\bullet} \leq \delta^{\bullet}$ by Proposition 2.12 (ii), whence as required. For transitivity, if $p_1 {\sim_{\delta_1}} p_2$ and $p_2 {\sim_{\delta_2}} p_3$, then for some q we have $F : p_1 \mapsto_{f_1} q$ and $F : p_2 \mapsto_{f_2} q$ and $F : p_3 \mapsto_{f_3} q$ as well as $f_2^{\bullet} \circ f_1 = \delta_1$ and $f_3^{\bullet} \circ f_2 = \delta_2$. But then:

$$\delta_2 \circ \delta_1 = f_3 \circ f_2 \circ f_2^{\bullet} \circ f_1 = f_3 \circ f_1$$

hence $p_1 {\sim_{\delta_2 \circ \delta_1}} p_3$. Then the general case of $\sim_F$ is proved as for the case for symmetry, using the fact $\circ$ commutes with $\leq$.

(ii) Easy from Propositions 2.9 (ii) and 3.18 (ii).

(iii) We construct the isomorphism I. Let $\eta_{\sim_F}$ be the natural map as constructed in (i). Let

$$I \stackrel{\mathrm{def}}{=} F \circ \eta_{\sim_F}^{-1}. \tag{5}$$

Clearly I is everywhere defined. Let $F : p \mapsto_f q$ and $\eta_{\sim_F} : p \mapsto_{f'} [p]_{\sim_F}$. We observe one essential fact: by Lemma 3.28, we know $\mathtt{pre}(f) = p| \sim_F$, while by definition we know $\mathtt{pre}(f') = p|\sim_F$. Thus, by Proposition 2.15 (i), we have $f \circ p|\sim_F = f$ and $f' \circ p|\sim_F = f'$. Now let $f'' = f \circ f'^{\bullet}$ (note f'' is a component of I). We now show f'' is a strong isomorphism, which gives the required result. But if $\rho \in \mathcal{S}([p]_{\sim_F})$ then, for $u \in \langle \sim_F |p\rangle$, we have

$$\begin{aligned} f'' \circ \rho &= f \circ u \circ f'^{\bullet} \\ &= f \circ u \circ (f^{\bullet} \circ f) \circ f'^{\bullet} \\ &= (f \circ u \circ f^{\bullet}) \circ f''. \end{aligned}$$

But $f \circ u \circ f^{\bullet} (= u' \circ f \circ f') \in \mathcal{S}(q)$, so f'' is symmetry preserving. By exactly the same reasoning, we can show $f''^{\bullet}$ is symmetry preserving. Moreover, using observation on $\mathtt{pre}(f)$ and $\mathtt{pre}(f')$ we gave above, we have:

$$\begin{aligned} f'' \circ f''^{\bullet} &= f \circ f'^{\bullet} \circ f' \circ f^{\bullet} \\ &= f \circ p|\sim_F \circ f^{\bullet} \\ &= f \circ f^{\bullet} \;=\; \mathbf{1}_p \end{aligned}$$

so that f'' is a map. Similarly we know $f''^{\bullet}$ is a map, establishing that f is a strong isomorphism, hence I is a p-isomorphism. By observing $\eta_{\sim_F}^{-1} \circ \eta_{\sim_F} = F^{-1} \circ F$ (note if $\langle p,\ f'^{\bullet} \circ f,\ q\rangle \in \sim_F$ then $\mathtt{pre}(f) = p|\sim_F$ and $\mathtt{pre}(f') = q|\sim_F$ which shows, if i and i' are their representations, we have $f'^{\bullet} \circ f = i'^{\bullet} \circ i$, hence the result) as well as $G \circ G^{-1} = \mathbf{ID}$ for any p-map G, the equation (5) can be rewritten:

$$I \circ \eta_{\sim_F} \;=\; F \circ \eta_{\sim_F}^{-1} \circ \eta_{\sim_F} \;=\; F \circ F^{-1} \circ F \;=\; F$$

as required. For uniqueness, let $F = I' \circ \eta_{\sim_F}$, multiply both sides from the right by $\eta_{\sim_F}^{-1}$, then $I' = I$, hence done. This establishes the statement. ■

3.29 Conclusion of Part One. We have so far shown that set-like theory can be developed consistently for process structure with an arbitrary base domain. It is notable that all the preceding (and subsequent, for that matter) results hold with any base domain: thus we have proved those results once and for all in terms of quite varied structures, some standard and others (currently) non-standard, which may be exploited for theory of computing. Another notable aspect is that the proofs in the present chapter is no more complex, and sometimes more lucid, than our preceding "concrete" developments. For example, the proof of Proposition 3.14 is strikingly much shorter than that in [9]. Moreover the abstract theory opens ways to think about concrete structures of processes beyond those which have been treated in the foregoing study of concurrency. The abstract theory will be made use of in a crucial way in Part Two of the present Notes, where we develop theory of abstract algebra on the basis of our development in Part One, and also in the study of process dynamics in the subsequent Notes.

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