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Abstract

This paper describes a generalisation of the adjunction and duality between topological spaces and frames, giving a natural concrete semantics to intuitionistic modal logic. A pre-modal frame is defined as a frame carrying two operations $\Box$ and $\Diamond$, and a topological relational structure is defined as a topological space with a binary relation on the points. With appropriate definitions of morphisms, there is a functor from the topological category to the algebraic, which gives a semantics for intuitionistic modal logic generalising the Kripke semantics of classical modal logic.

This paper describes an adjoint to the semantic functor, constructing the universal topological relational structure on a pre-modal frame. The adjunction restricts to a duality between the images of the two functors. Finally the spectral (or coherent) case where the frame is isomorphic to the frame of ideals of a distributive lattice is shown to be in the scope of the duality. This gives a completeness result for four simple axioms for the topological semantics.

1 Introduction

Classical modal logic is the study of necessity and possibility (and related constructs) in the context of a Boolean propositional language. Before the work of Kripke [4], research in this area revolved around philosophical arguments about the logical axioms the modalities should satisfy. By providing a universal semantics applying to all normal modal systems, Kripke translated such problems into straightforward properties of binary relations. So well does the Kripke semantics capture the intuitive meaning of modal connectives, that some logicians now consider modal logic to be nothing more than a calculus for studying binary relations (see [6], for instance).

What makes the Kripke semantics so powerful is the equivalence, or rather *duality*, between straightforward algebraic models of modal logic—the modal algebras—and relational structures. The soundness result shows that the powerset of a relational structure has a natural modal algebra structure, and the completeness proof shows that the maximal consistent sets of a modal algebra have a natural relational structure. The right definitions of morphism between modal algebras and between relational structures mean that each class is a category, and that the correspondence generalises to a contravariant equivalence between them. It is this mathematical result which lies behind the powerful results of correspondence theory: a relational structure satisfies some property if, and only if, the corresponding modal algebra satisfies some axiom.

Intuitionistic modal logic is currently in a state similar to classical modal logic before Kripke. There is no generally accepted semantic framework, and there is considerable disagreement on the basic axioms. Most developments are based on combining Kripke's semantics of modal logic with his (closely related) semantics of intuitionistic logic [5]. This leads to models with two binary relations, one of which is a pre-order. However, there is still no agreement on what the relationship between the two relations should be (see [7] for a discussion of several possibilities), and it seems that this approach does not provide the kind of concrete semantics which appeals intuitively.

In this paper we take a rather different approach, emphasising the duality, and taking our intuition from natural, concrete models. Instead of trying to combine the two Kripke semantics, we combine two dualities: that between modal algebras and relational structures discussed above, and that between *frames* (complete Heyting algebras) and *topological spaces* [2].

To motivate this approach, consider a naïve model of temporal logic: the real numbers with their usual ordering. This is the notion of time used by (Newtonian) physics, and all temporal logics are, at least implicitly, related to it. However, as a relational model of classical temporal logic, it has some counter-intuitive features. Since propositions are modelled by arbitrary subsets, the model includes propositions which are true for a single instant; it distinguishes between propositions which differ only on a null set; it even includes propositions which are true only on an unmeasurable set. These features are clearly not practically observable, if they have any physical meaning at all.

In order to solve this problem, we consider the (intuitively reasonable) proposal that, in order for a proposition to be true at time t, it must be continuously true for some interval $(t-\delta, t+\delta)$ about t. This means that propositions are modelled by the *open* sets in the usual topology, and that the logic is intuitionistic.

Generalising this example to other intuitionistic modal logics leads us to consider relational structures with a topology on the elements, and one of the results of this paper is a sound and complete axiomatisation of this model. By forgetting the relation, we recover the usual topological semantics of intuitionistic logic; by using the discrete topology we recover the usual Kripke semantics of classical modal logic. However, our *approach* owes much more to the topological tradition than to that of classical logic. Rather than simply proving soundness and completeness results, we wish also to consider morphisms between structures (generalising the *p-morphisms*). We therefore define categories of algebraic and topological structures, construct a contravariant adjunction between them, and show when this adjunction restricts to an equivalence.

The construction of the adjoint functor from the algebraic to the topological category is a significant part of the mathematical contribution of this paper. Although the algebraic structures are frames (with extra structure) the usual notion of *points* does not make enough distinctions to construct the universal topological space. Instead we consider certain *trees* of frame points, and define the accessible worlds as the immediate subtrees. This definition successfully constructs the adjoint, but means that the intuitionistic case is rather more complex that the classical (where the worlds are the same in the modal and non-modal cases).

This paper combines material from two rather different fields, and it has been necessary to assume a lot of knowledge of both of them. Our terminology for duality theory is taken from [2], and for modal logic from [6]. Any concepts which are unfamiliar to the reader should be found in one or other of these books.

One final point of terminology must be made: in this paper a **frame** is a partially ordered set with finite meets, all joins and the distributivity law. A set (of worlds) with a binary relation (of accessibility) is called a **relational structure**. The author hopes that those who are used to the term "Kripke frame" will not suffer greatly from culture shock.

2 The Algebraic Categories

In this section we consider the intuitionistic analogues of modal algebras. We cannot immediately write down a straightforward definition for two reasons. Firstly, we wish to consider several different ordered structures, including distributive lattices, Heyting algebras and frames. Secondly, there is much disagreement on the axiomatisation of intuitionistic modal logic, and each axiom is another condition on the corresponding notion of algebra. Fortunately, the constructions we are interested in work on a very general notion of modality, and we can sidestep the question of what the "right" set of axioms is by working at this level of generality. This is not to undermine the importance of axiomatising the most natural and useful structures: we use the term "pre-modal" in order to save the term "modal" for these. In the author's opinion, this paper provides evidence that the "right" axioms are (1–4) below.

Definition. Let $(A, \leqslant)$ be a partially ordered set. A **pre-modality** on A is a pair of monotone maps $\Box, \Diamond : A \to A$.

A **pre-modal** distributive lattice, Heyting algebra or frame is a distributive lattice, Heyting algebra or frame (respectively) with a distinguished pre-modality.

The definition of "pre-modality" is chosen to be the most general which makes the proofs work, rather than the most useful. What is missing is a set of axioms to take the place of K in classical modal logic. The following additional axioms (or other equivalent ones) have been used by various authors; some, at least, of these should be satisfied by a structure before it deserves the term "modality." They are collected together here for easy reference in the rest of the paper.

$$
\begin{align}
\top \leqslant \Box\top \tag{1}\\
\Box a \wedge \Box b \leqslant \Box(a \wedge b) \tag{2}\\
\Box a \wedge \Diamond b \leqslant \Diamond(a \wedge b) \tag{3}\\
\Diamond\bot \leqslant \bot \tag{4}\\
\Diamond(a \vee b) \leqslant \Diamond a \vee \Diamond b \tag{5}\\
\Box(a \vee b) \leqslant \Box a \vee \Diamond b \tag{6}\\
\Diamond a \to \Box b \leqslant \Box(a \to b) \tag{7}\\
\Diamond a \to \Box b \leqslant \Box(a \vee b) \to \Box b \tag{8}
\end{align}
$$

Axioms (1–4) are agreed by most authors; [8] proves soundness and completeness for a certain class of (Kripke-style) structures. Axioms (1–6) are used in [3] for the construction of Vietoris locales, and in [1] for the construction of powerdomains. Axioms (1–5,7) are championed by [7], who proves soundness and completeness for internal models in a topos. Axiom (8) appears to be new; note that it follows both from (2, 7) and from (6).

Of course the definition of a pre-modality is so general that examples are numerous, and we do not wish to burden the reader with large numbers of pathological and unilluminating cases. The important examples from the logical point of view are the pre-modal Heyting algebras constructed as Lindenbaum algebras of modal logical calculi. Which of the above axioms are satisfied by such examples depends entirely on which of the corresponding propositions are provable in the logic.

The following proposition allows us to complete distributive lattices and Heyting algebras to frames, in most cases.

Proposition 1. Let $(A, \leqslant, \Box, \Diamond)$ be a pre-modal distributive lattice, and $\mathrm{Idl}(A)$ its frame of ideals. The pre-modality on A lifts to one on $\mathrm{Idl}(A)$ as follows:

$$
\begin{aligned}
\Box(I) &= \{a | \exists b \in I.\ a \leqslant \Box b\}\\
\Diamond(I) &= \{a | \exists b \in I.\ a \leqslant \Diamond b\}
\end{aligned}
$$

Furthermore, if A satisfies any of the axioms (1–6,8), so does $\mathrm{Idl}(A)$.

Proof. That $\mathrm{Idl}(A)$ is a pre-modal frame is immediate; we leave checking each of the axioms as an easy exercise for the reader. □

Axiom (7) is conspicuous by its absence from proposition 1. The following counterexample shows that this axiom is not preserved by the ideal construction.

Let $A = \mathcal{P}(\mathbb{N})$, with the inclusion ordering, and let

$$
\Box(a) = \begin{cases} \mathbb{N} & \text{if } a \text{ is cofinite} \\ \emptyset & \text{otherwise} \end{cases}
$$

$$
\Diamond(a) = \begin{cases} \emptyset & \text{if } a \text{ is finite} \\ \mathbb{N} & \text{otherwise} \end{cases}
$$

Then A is a Boolean algebra, $\Box$ preserves finite meets, and $\Diamond(a) = \neg\Box(\neg a)$, so this pre-modal distributive lattice satisfies all the axioms (1–8).

Now consider the ideals

$$
\begin{aligned}
I &= \{a \in A | a \text{ is finite}\}\\
J &= \{\emptyset\}
\end{aligned}
$$

Then $\Diamond(I) \to \Box(J) = A$ but $\Box(I \to J) = \emptyset$ so axiom (7) fails in $\mathrm{Idl}(A)$.

In order to define categories, we have to say what the arrows are. Care is needed here, as the most useful notion of morphism is not the algebraic one, but a 'lax' version of it.

Definition. Let $(A, \leqslant, \Box, \Diamond)$ and $(B, \leqslant', \Box', \Diamond')$ be pre-modal posets. A **modal morphism** $f : (A, \leqslant, \Box, \Diamond) \to (B, \leqslant', \Box', \Diamond')$ is a monotone map $f : (A, \leqslant) \to (B, \leqslant')$ satisfying

$$f(\Box a) \leqslant' \Box' f(a) \tag{9}$$
$$f(\Diamond a) \leqslant' \Diamond' f(a) \tag{10}$$

If $(A, \leqslant)$ and $(B, \leqslant')$ are distributive lattices, Heyting algebras, or frames and f is the corresponding kind of morphism, we call it a **modal distributive-lattice morphism**, **modal Heyting-algebra morphism** or **modal frame morphism** respectively.

If the inequalities (9) and (10) are strengthened to equalities, we say f is **strict**.

This gives us three important categories:

- PMDLat is the category of pre-modal distributive lattices and modal distributive-lattice morphisms;
- PMHeyt is the category of pre-modal Heyting algebras and modal Heyting-algebra morphisms;
- PMFrm is the category of pre-modal frames and modal frame morphisms.

An example of a modal distributive-lattice morphism is given by the embedding of a pre-modal distributive lattice in its ideal completion. Because this map is strict and one-one, no structure is lost in passing to the frame of ideals.

Lemma 2. Let $(A, \leqslant, \Box, \Diamond)$ be a pre-modal distributive lattice, and $(\mathrm{Idl}(A), \subseteq, \Box', \Diamond')$ its pre-modal frame of ideals, as defined in proposition 1. Then $f_A : (A, \leqslant, \Box, \Diamond) \to (\mathrm{Idl}(A), \subseteq, \Box', \Diamond')$ defined by

$$f_A(a) = \{b | b \leqslant a\}$$

is a strict, one-one, modal distributive-lattice morphism.

Furthermore, if $(A, \leqslant)$ is a Heyting algebra, then f_A is a modal Heyting-algebra morphism.

Proof. That f_A is a one-one distributive-lattice/Heyting-algebra morphism is standard. Now

$$\begin{aligned} b \in \Box' f_A(a) &\iff \exists c \in f_A(a).\ b \leqslant \Box c \\ &\iff \exists c \leqslant a.\ b \leqslant \Box c \\ &\iff b \leqslant \Box a \\ &\iff b \in f_A(\Box a) \end{aligned}$$

so $\Box' f_A(a) = f_A(\Box a)$; similarly, $\Diamond' f_A(a) = f_A(\Diamond a)$ so f_A is a strict modal morphism. ∎

We can characterise $\mathrm{Idl}(A)$ (up to isomorphism) as the free pre-modal frame built from A: the map f_A is universal from A to the forgetful functor from pre-modal frames. The details are in the following theorem.

Theorem 1. The construction of proposition 1 defines a functor

$$\mathcal{I} : \mathrm{PMDLat} \to \mathrm{PMFrm}$$

left adjoint to the forgetful functor, and f_A defined in lemma 2 is the unit of the adjunction.

Proof. Let $g : (A, \leqslant, \Box, \Diamond) \to (B, \leqslant', \Box', \Diamond')$ where A is a distributive lattice and B is a frame. Define $g^\sharp : \mathrm{Idl}(A) \to B$ by

$$g^\sharp(I) = \bigvee \{g(a) | a \in I\}$$

It is standard that this is the unique frame morphism satisfying $g^\sharp f_A = g$. Now

$$\forall a \in I.\ g(a) \leqslant \bigvee\{g(a)|a \in I\}$$
$$\forall a \in I.\ \Box' g(a) \leqslant \Box' \bigvee\{g(a)|a \in I\}$$
$$\forall a \in I.\ g(\Box a) \leqslant \Box' \bigvee\{g(a)|a \in I\}$$
$$\forall b.\ (\exists a \in I.\ b \leqslant \Box a) \Rightarrow g(b) \leqslant \Box' \bigvee\{g(a)|a \in I\}$$
$$\bigvee\{g(b)|\exists a \in I.\ b \leqslant \Box a\} \leqslant \Box' \bigvee\{g(a)|a \in I\}$$

so $g^\sharp(\Box I) \leqslant \Box' g^\sharp(I)$ and similarly $g^\sharp(\Diamond I) \leqslant \Diamond' g^\sharp(I)$. Therefore $g^\sharp$ is a modal distributive-lattice morphism. $\Box$

The categories PMDLat, PMHeyt and PMFrm have a lot of categorical structure, including various completeness, cocompleteness and factorisation properties. However, this is outside the scope of this paper, and the relevance of such results to applications is not yet clear. We therefore leave this subject and turn to the topological models.

3 Semantics

In this section we define the category of topological models which we wish to study, give our semantics of intuitionistic modal logic, and discuss the soundness of the axioms (1–8).

Definition. A **topological relational structure** is a topological space $(X, \mathfrak{O})$ together with a binary relation $\mathrm{R} \subseteq X \times X$.

Note that we have not asserted any relationship between the accessibility relation and the topology. Although there are several such natural conditions (some of which are considered below), each of them has important counterexamples, which we do not wish to exclude.

The notion of topological relational structure is a general but natural one, occurring throughout mathematics. The first two of the following examples show how to include the familiar cases of classical modal logic and non-modal intuitionistic logic. The next three are the temporal model mentioned in the introduction, and two generalisations of it. The final example is the graph of a discontinuous function.

1. Any relational structure/one-label transition system with the discrete topology.
2. Any topological space with the empty relation, the diagonal relation or the total relation.
3. The real numbers $\mathbb{R}$ with their usual topology and their usual ordering $\leqslant$.
4. The set

   $$\{(x, y) \in \mathbb{R}^2 | x \leqslant 0 \ \&\ y = 0 \text{ or } x \geqslant 0 \ \&\ y = \pm x\}$$

   with the topology inherited from $\mathbb{R}^2$ and the order defined by

   $$(x, y) \leqslant (x', y') \iff x \leqslant x' \ \&\ yy' \geqslant 0$$

   This gives a simple model of branching time (with one branch point at $x = 0$).
5. Any relativistic space time with the relation $x \,\mathrm{R}\, y$ iff y is inside the light cone of x.
6. The real numbers $\mathbb{R}$ with their usual topology and the relation defined by

   $$x \,\mathrm{R}\, {-1} \iff x < 0$$
   $$x \,\mathrm{R}\, 0 \iff x = 0$$
   $$x \,\mathrm{R}\, 1 \iff x > 0$$

   and no other related pairs.

Some notation is useful when investigating topological relational structures: we define $\mathrm{R}^{\blacksquare}, \mathrm{R}^{\blacklozenge}, \mathrm{R}^{\bullet} : \mathcal{P}X \to \mathcal{P}X$ by

$$\begin{aligned} \mathrm{R}^{\blacksquare}(U) &= \{x \in X | \forall y \in X.\ x \,\mathrm{R}\, y \Rightarrow y \in U\} \\ \mathrm{R}^{\blacklozenge}(U) &= \{x \in X | \exists y \in X.\ x \,\mathrm{R}\, y \;\&\; y \in U\} \\ \mathrm{R}^{\bullet}(U) &= \{y \in X | \exists x \in X.\ x \,\mathrm{R}\, y \;\&\; x \in U\} \end{aligned}$$

The maps $\mathrm{R}^{\blacksquare}$ and $\mathrm{R}^{\blacklozenge}$ are the familiar $\Box$ and $\Diamond$ of classical modal logic, and $\mathrm{R}^{\bullet}$ is the direct image map. However, they do not preserve open sets (except in special cases) so we need to take the interior in order to give the semantics of intuitionistic modal logic.

Definition. If $(X, \mathfrak{O}, \mathrm{R})$ is a topological relational structure, we define a pre-modality $\Box^{\mathrm{R}}, \Diamond^{\mathrm{R}}$ on the frame $(\mathfrak{O}, \subseteq)$ of open sets by

$$\begin{aligned} \Box^{\mathrm{R}}(U) &= (\mathrm{R}^{\blacksquare}(U))^{\circ} \\ \Diamond^{\mathrm{R}}(U) &= (\mathrm{R}^{\blacklozenge}(U))^{\circ} \end{aligned}$$

If $(A, \leqslant, \Box, \Diamond)$ is a pre-modal distributive lattice/Heyting algebra/frame, a **model** of A is a topological relational structure $(X, \mathfrak{O}, \mathrm{R})$ together with a strict modal distributive-lattice/Heyting-algebra/frame morphism (respectively) $f : (A, \leqslant, \Box, \Diamond) \to (\mathfrak{O}, \subseteq, \Box^{\mathrm{R}}, \Diamond^{\mathrm{R}})$.

If A is defined as the Lindenbaum algebra of a propositional modal calculus, any function from propositional variables to open sets can be extended to a model in the familiar way, provided the axioms of the calculus are true of the topological space—i.e. sound in the model. The following proposition gives sufficient conditions for the soundness of the axioms listed in section 2.

Proposition 3. Let $(X, \mathfrak{O}, \mathrm{R})$ be a topological relational structure, and $(\mathfrak{O}, \subseteq, \Box^{\mathrm{R}}, \Diamond^{\mathrm{R}})$ the corresponding pre-modal frame. Then

1. $(\mathfrak{O}, \subseteq, \Box^{\mathrm{R}}, \Diamond^{\mathrm{R}})$ satisfies axioms (1–4);
2. if $\mathrm{R}^{\blacklozenge}$ preserves open sets then $(\mathfrak{O}, \subseteq, \Box^{\mathrm{R}}, \Diamond^{\mathrm{R}})$ satisfies axioms (5,8);
3. if $\mathrm{R}^{\blacklozenge}$ and $\mathrm{R}^{\blacksquare}$ preserve open sets then $(\mathfrak{O}, \subseteq, \Box^{\mathrm{R}}, \Diamond^{\mathrm{R}})$ satisfies axiom (6);
4. if $\mathrm{R}^{\blacklozenge}$ and $\mathrm{R}^{\bullet}$ preserve open sets then $(\mathfrak{O}, \subseteq, \Box^{\mathrm{R}}, \Diamond^{\mathrm{R}})$ satisfies axiom (7).

Proof.

1. These four follow from the corresponding properties of $\mathrm{R}^{\blacksquare}$ and $\mathrm{R}^{\blacklozenge}$ (familiar from the classical case) together with the fact that interior preserves finite intersections and the empty set.
2. If $\mathrm{R}^{\blacklozenge}$ preserves open sets then $\Diamond^{\mathrm{R}} = \mathrm{R}^{\blacklozenge}$ so (5) follows from the classical case. For (8), we have

$$\begin{aligned} \Box(a \vee b) \wedge (\Diamond a \to \Box b) &\leqslant \mathrm{R}^{\blacksquare}(a \vee b) \wedge (\mathrm{R}^{\blacklozenge}\, a \to \mathrm{R}^{\blacksquare}\, b) \\ &\leqslant \mathrm{R}^{\blacksquare}\, b \end{aligned}$$

   but the set on the left is open, so the result follows.
3. If $\mathrm{R}^{\blacksquare}$ preserves open sets then $\Box^{\mathrm{R}} = \mathrm{R}^{\blacksquare}$ so this follows from the classical case.
4. Let $y \in \mathrm{R}^{\bullet}(\Diamond^{\mathrm{R}}(U) \to \Box^{\mathrm{R}}(V)) \cap U$. By definition of $\mathrm{R}^{\bullet}$, there exists $x \in X$ such that $x \,\mathrm{R}\, y$ and $x \in \Diamond^{\mathrm{R}}(U) \to \Box^{\mathrm{R}}(V)$, but $y \in U$ so $x \in \mathrm{R}^{\blacklozenge}(U)$ by definition. We assumed that $\mathrm{R}^{\blacklozenge}(U) = \Diamond^{\mathrm{R}}(U)$, so $x \in \Box^{\mathrm{R}}(V) \subseteq \mathrm{R}^{\blacksquare}(V)$, which implies that $y \in V$.

   We have shown that

$$\mathrm{R}^{\bullet}(\Diamond^{\mathrm{R}}(U) \to \Box^{\mathrm{R}}(V)) \cap U \subseteq V$$

so, since $\mathrm{R}^{\bullet}$ preserves open sets

$$\mathrm{R}^{\bullet}(\Diamond^{\mathrm{R}}(U) \to \Box^{\mathrm{R}}(V)) \subseteq U \to V$$

so (as in the classical case)

$$\Diamond^{\mathrm{R}}(U) \to \Box^{\mathrm{R}}(V) \subseteq \mathrm{R}^{\blacksquare}(U \to V)$$

but the set on the left is open, so

$$\Diamond^{\mathrm{R}}(U) \to \Box^{\mathrm{R}}(V) \subseteq \Box^{\mathrm{R}}(U \to V)$$

as required.

$\Box$

The three "continuity" conditions that $\mathrm{R}^{\blacksquare}$, $\mathrm{R}^{\blacklozenge}$ and $\mathrm{R}^{\bullet}$ preserve open sets are fairly natural conditions on a topological relation. In the case where R is the graph of a function, the first two are equivalent and say that the function is continuous; the third one says that the function is open. However, these conditions are not true in all cases of interest, and the examples at the beginning of this section provide counterexamples to each of the axioms (5–8).

Axiom (5) fails in example 4: let X be the space of example 4, and

$$U = \{(x, y) \in X | x > 0 \;\&\; y = x\}$$
$$V = \{(x, y) \in X | x > 0 \;\&\; y = -x\}$$

then

$$\Diamond(U \cup V) = X$$
$$\Diamond U \cup \Diamond V = X \setminus \{(0, 0)\}$$

So $(0, 0)$ is in the first set but not the second.

Axiom (6) fails in the same example: let X and V be as before, and

$$W = \{(x, y) \in X || x| < 1 \text{ or } y > 0\}$$

then

$$\Box(V \cup W) = \{(x, y) \in X | x > -1\}$$
$$\Diamond V \cup \Box W = X \setminus \{(0, 0)\}$$

As before, $(0, 0)$ is in the first set but not in the second.

Axiom (7) fails in example 6: let

$$U = (-\tfrac{1}{2}, 1\tfrac{1}{2})$$
$$V = (\tfrac{1}{2}, 1\tfrac{1}{2})$$

then

$$\Diamond U \to \Box V = \mathbb{R}$$
$$\Box(U \to V) = \mathbb{R} \setminus \{0\}$$

Again, 0 is in the first set but not in the second.

Axiom (8) fails in the same example: let X and U be as before, and

$$W = (-1\tfrac{1}{2}, -\tfrac{1}{2}) \cup (\tfrac{1}{2}, 1\tfrac{1}{2})$$

then

$$\begin{aligned}\Box(U\cup W)&=\mathbb{R}\\ \Diamond U\to\Box W&=\mathbb{R}\\ \Box W&=\mathbb{R}\setminus\{0\}\end{aligned}$$

This time 0 is in the first two sets but not in the third.

In order to get a useful category of topological relational structures, we need the right definition of morphism. The following is a generalisation of the notion of "p-morphism" used in modal logic.

Definition. Let $(X,\mathfrak{O},\mathrm{R})$ and $(Y,\mathfrak{P},S)$ be topological relational structures. A **continuous relational morphism** $f:(X,\mathfrak{O},\mathrm{R})\to(Y,\mathfrak{P},S)$ is a continuous function $f:(X,\mathfrak{O})\to(Y,\mathfrak{P})$ satisfying

$$x\,\mathrm{R}\,y\Rightarrow f(x)Sf(y) \tag{11}$$

$$U\in\mathfrak{P}\ \&\ \exists y\in U.\ f(x)Sy\Rightarrow\exists x'\in X.\ x\,\mathrm{R}\,x'\ \&\ f(x')\in U \tag{12}$$

The category of topological relational structures and continuous relational morphisms we denote TRS.

Condition (12) is a generalisation of the "back and forth" property of p-morphisms: if the topology is discrete, we recover the usual property by taking $U=\{y\}$. In other topologies this definition is considerably more general than just adding the continuity condition to the usual definition of p-morphism. That it is not too general is shown by the following result.

Proposition 4. If $f:(X,\mathfrak{O},\mathrm{R})\to(Y,\mathfrak{P},S)$ is a continuous relational morphism then $f^{\leftarrow}:(\mathfrak{P},\subseteq,\Box^S,\Diamond^S)\to(\mathfrak{O},\subseteq,\Box^{\mathrm{R}},\Diamond^{\mathrm{R}})$ is a modal frame morphism.

The mapping

$$\begin{aligned}\mathcal{O}&:\mathrm{TRS}\to\mathrm{PMFrm}\\ \mathcal{O}(X,\mathfrak{O},\mathrm{R})&=(\mathfrak{O},\subseteq,\Box^{\mathrm{R}},\Diamond^{\mathrm{R}})\\ \mathcal{O}(f)&=f^{\leftarrow}\end{aligned}$$

is a contravariant functor.

Proof. That $f^{\leftarrow}$ is a frame morphism follows from continuity. It remains to show that $f^{\leftarrow}(\Box^S U)\subseteq\Box^{\mathrm{R}}f^{\leftarrow}(U)$ and $f^{\leftarrow}(\Diamond^S U)\subseteq\Diamond^{\mathrm{R}}f^{\leftarrow}(U)$.

If $x\in f^{\leftarrow}(\Box^S U)$ then $f(x)\in\Box^S(U)\subseteq S^{\blacksquare}(U)$ i.e.

$$\forall y\in Y.\ f(x)Sy\Rightarrow y\in U$$

Let $x'\in X$ s.t. $x\,\mathrm{R}\,x'$. Then $f(x)Sf(x')$ by (11), so $f(x')\in U$, i.e. $x'\in f^{\leftarrow}(U)$. Therefore $f^{\leftarrow}(\Box^S U)\subseteq\mathrm{R}^{\blacksquare}f^{\leftarrow}(U)$; but $f^{\leftarrow}(\Box^S U)$ is open (because f is continuous) so $f^{\leftarrow}(\Box^S U)\subseteq\Box^{\mathrm{R}}f^{\leftarrow}(U)$.

If $x\in f^{\leftarrow}(\Diamond^S U)$ then $f(x)\in\Diamond^S(U)\subseteq S^{\blacklozenge}(U)$ i.e.

$$\exists y\in Y.\ f(x)Sy\ \&\ y\in U$$

So by (12),

$$\exists x'\in X.\ x\,\mathrm{R}\,x'\ \&\ f(x')\in U$$

Therefore $f^{\leftarrow}(\Diamond^S U)\subseteq\mathrm{R}^{\blacklozenge}f^{\leftarrow}(U)$; but $f^{\leftarrow}(\Diamond^S U)$ is open, so $f^{\leftarrow}(\Diamond^S U)\subseteq\Diamond^{\mathrm{R}}f^{\leftarrow}(U)$.

The functoriality of $\mathcal{O}$ is immediate. $\Box$

4 The Adjunction

In this section we construct a topological relational structure from any pre-modal frame, in a universal way. In other words, we define an adjoint to the functor $\mathcal{O}$. We go on to show that this adjunction restricts to a duality between the images of the functors.

Let $(A, \leqslant, \Box, \Diamond)$ be a pre-modal frame. Recall that a **completely prime filter** of A is a set $p \subseteq A$ satisfying

$$a \in p \ \&\ a \leqslant b \Rightarrow b \in p$$
$$\top \in p$$
$$a \in p \ \&\ b \in p \Rightarrow a \wedge b \in p$$
$$\bigvee S \in p \Rightarrow \exists a \in S.\ a \in p$$

A completely prime filter is sometimes called a point of A, but we shall not use this term, as these are not the points of the topological relational structure we wish to construct.

We wish to consider certain arbitrarily branching trees of completely prime filters. It is convenient to represent such a tree as the set of finite paths through it, starting at the root.

Definition. A **tree** of A is a non-empty set t of non-empty finite sequences of completely prime filters of A, satisfying

$$\sigma\tau \in t \ \&\ \sigma \neq \langle\rangle \Rightarrow \sigma \in t \tag{13}$$
$$\langle p\rangle \in t \ \&\ \langle q\rangle \in t \Rightarrow p = q \tag{14}$$
$$\sigma\langle p, q\rangle \in t \ \&\ \Box a \in p \Rightarrow a \in q \tag{15}$$
$$\sigma\langle p\rangle \in t \ \&\ \Diamond a \in p \Rightarrow \exists q.\ \sigma\langle p, q\rangle \in t \ \&\ a \in q \tag{16}$$

The set of all trees of A is denoted $\mathrm{tree}(A)$. We let greek letters σ, τ range over (possibly empty) sequences of completely prime filters.

The **root** of $t \in \mathrm{tree}(A)$ is the unique prime filter $\mathrm{rt}(t)$ satisfying $\langle \mathrm{rt}(t)\rangle \in t$.

The following lemma shows that the elements of $\mathrm{tree}(A)$ really are trees.

Lemma 5. If t is a tree and $\langle \mathrm{rt}(t), p\rangle \in t$, then the immediate subtree

$$t(p) = \{\langle p\rangle\sigma | \langle \mathrm{rt}(t), p\rangle\sigma \in t\}$$

is a tree.

Proof. (13) If $\sigma\tau \in t(p)$ and $\sigma \neq \langle\rangle$ then $\sigma\tau = \langle p\rangle\sigma'\tau$ and $\langle \mathrm{rt}(t), p\rangle\sigma'\tau \in t$ so $\langle \mathrm{rt}(t), p\rangle\sigma' \in t$ so $\sigma = \langle p\rangle\sigma' \in t(p)$.

(14) If $\langle p'\rangle \in t(p)$ then $p' = p$.

(15) If $\sigma\langle q, r\rangle \in t(p)$ and $\Box b \in q$ then $\langle \mathrm{rt}(t)\rangle\sigma\langle q, r\rangle \in t$ so $b \in r$.

(16) If $\sigma\langle q\rangle \in t(p)$ and $\Diamond b \in q$ then $\langle \mathrm{rt}(t)\rangle\sigma\langle q\rangle \in t$ so there is a prime filter r such that $b \in r$ and $\langle \mathrm{rt}(t)\rangle\sigma\langle q, r\rangle \in t$, so $\sigma\langle q, r\rangle \in t(p)$. $\Box$

The trees $t \in \mathrm{tree}(A)$ are the possible traces through relations, starting at points which satisfy the propositions in $\mathrm{rt}(t)$, and consistent with the semantics of $\Box$ and $\Diamond$. In order to construct the universal topological relational structure, it is necessary to identify those trees whose subtrees are indistinguishable. This is achieved by the following definition.

Definition. Two trees t and u are **equivalent** if they are related by $t \sim u$:

$$\begin{aligned} t \sim u \iff & \mathrm{rt}(t) = \mathrm{rt}(u) \\ & \&\ \forall p.\ \langle \mathrm{rt}(t), p\rangle \in t \Rightarrow \forall a \in p.\ \exists q \ni a.\ \langle \mathrm{rt}(u), q\rangle \in u \\ & \&\ \forall q.\ \langle \mathrm{rt}(u), q\rangle \in u \Rightarrow \forall a \in q.\ \exists p \ni a.\ \langle \mathrm{rt}(t), p\rangle \in t \end{aligned}$$

After lemma 6, a **tree point** is an equivalence class of trees. We write the equivalence class of t as $[t]$, and the set of tree points of A as $\mathrm{tp}(A)$.

Lemma 6. The relation $\sim$ is an equivalence relation on $\text{tree}(A)$.

Proof. Reflexivity and symmetry are immediate. For transitivity, assume $t \sim u$ and $u \sim v$.

Certainly $\text{rt}(t) = \text{rt}(u) = \text{rt}(v)$. If $\langle \text{rt}(t), p\rangle \in t$ and $a \in p$, then there is a $q \ni a$ such that $\langle \text{rt}(u), q\rangle \in u$, so there is an $r \ni a$ such that $\langle \text{rt}(v), r\rangle \in v$. The other condition is similar. $\square$

The tree points of A are the points of the topological relational structure universally associated with A. The following lemma shows that, just as in the non-modal case, this set has a natural topology.

Lemma 7. The map $\phi_A : A \to \mathcal{P}(\text{tp}(A))$ defined by

$$\phi_A(a) = \{[t] \in \text{tp}(A) | a \in \text{rt}(t)\}$$

is a frame morphism; its image is therefore a topology.

Proof. Straightforward calculation, exactly following the non-modal case. $\square$

Definition. The **topology** $\mathfrak{A}$ on $\text{tp}(A)$ is the image of ϕ_A.

The **relation** $\mathrm{R}_A \subseteq \text{tp}(A) \times \text{tp}(A)$ is defined by

$$x \, \mathrm{R}_A \, y \iff \exists t \in \text{tree}(A).\ x = [t] \;\&\; \exists p.\ \langle \text{rt}(t), p\rangle \in t \;\&\; y = [t(p)]$$

For each pre-modal frame $(A, \leqslant, \Box, \Diamond)$ we define the topological relational structure $\mathcal{G}(A)$ by

$$\mathcal{G}(A, \leqslant, \Box, \Diamond) = (\text{tp}(A), \mathfrak{A}, \mathrm{R}_A)$$

The unit of the adjunction is the map ϕ, which we show to be a modal morphism.

Proposition 8. The map $\phi_A : (A, \leqslant, \Box, \Diamond) \to (\mathfrak{A}, \subseteq, \Box^{\mathrm{R}_A}, \Diamond^{\mathrm{R}_A})$ of lemma 7 is a modal frame morphism.

Proof. That ϕ_A is a frame morphism follows immediately from lemma 7; it remains to show that $\phi_A(\Box a) \subseteq \Box^{\mathrm{R}_A}\phi_A(a)$ and $\phi_A(\Diamond a) \subseteq \Diamond^{\mathrm{R}_A}\phi_A(a)$.

Let $[t] \in \phi_A(\Box a)$ so $\Box a \in \text{rt}(t)$ by definition of ϕ_A. If $[t]\,\mathrm{R}_A[u]$ then there is a tree $t' \sim t$ satisfying $\langle \text{rt}(t), \text{rt}(u)\rangle \in t'$, so $a \in \text{rt}(u)$ by condition (15), i.e. $u \in \phi_A(a)$. Therefore $\phi_A(\Box a) \subseteq \mathrm{R}_A^{\blacksquare}\,\phi_A(a)$; since $\phi_A(\Box a)$ is open, this means that $\phi_A(\Box a) \subseteq \Box^{\mathrm{R}_A}\phi_A(a)$.

Let $[t] \in \phi_A(\Diamond a)$ so $\Diamond a \in \text{rt}(t)$. By condition (16), there is a completely prime filter p such that $\langle \text{rt}(t), p\rangle \in t$ and $a \in p$. It is immediate that $t\,\mathrm{R}_A\,t(p)$, and $a \in p = \text{rt}(t(p))$ so $\phi_A(\Diamond a) \subseteq \mathrm{R}_A^{\blacklozenge}\,\phi_A(a)$. Since $\phi_A(\Diamond a)$ is open, $\phi_A(\Diamond a) \subseteq \Diamond^{\mathrm{R}_A}\phi_A(a)$. $\square$

We now have all the ingredients of the adjunction.

Theorem 2. For any pre-modal frame $(A, \leqslant, \Box, \Diamond)$, the topological relational structure $(\text{tp}(A), \mathfrak{A}, \mathrm{R}_A)$ and the map ϕ_A give the reflection along $\mathcal{O}$. In other words, $\mathcal{G}$ extends uniquely to a contravariant functor adjoint to $\mathcal{O}$, with unit ϕ_A.

Proof. Let $(X, \mathfrak{O}, \mathrm{R})$ be a topological relational structure, and $f : (A, \leqslant, \Box, \Diamond) \to (\mathfrak{O}, \subseteq, \Box^{\mathrm{R}}, \Diamond^{\mathrm{R}})$ a modal frame morphism. We need to show the existence of a unique continuous relational morphism $f^\sharp : (X, \mathfrak{O}, \mathrm{R}) \to (\text{tp}(A), \mathfrak{A}, \mathrm{R}_A)$ satisfying $(f^\sharp)^{\leftarrow}\phi_A = f$.

First we prove existence. Define

$$\begin{aligned} f^\natural &: (X, \mathfrak{O}, \mathrm{R}) \to \text{tree}(A) \\ f^\natural(x) &= \{\langle f^*(x_1), \ldots, f^*(x_n)\rangle | x_1 = x,\ x_j \,\mathrm{R}\, x_{j+1}\} \\ f^\sharp &: (X, \mathfrak{O}, \mathrm{R}) \to (\text{tp}(A), \mathfrak{A}, \mathrm{R}_A) \\ f^\sharp(x) &= [f^\natural(x)] \end{aligned}$$

where $f^*(x) = \{a \in A | x \in f(a)\}$.

We claim that $f^\natural(x) \in \mathrm{tree}(A)$, for all $x \in X$. That $f^*(x)$ is a prime filter is exactly like the non-modal case, and conditions (13) and (14) are immediate.

(15) If $\sigma\langle f^*(y), f^*(z)\rangle \in f^\natural(x)$ and $\Box a \in f^*(y)$ then

$$y \in f(\Box a) \subseteq \Box^{\mathrm{R}} f(a) \subseteq \mathrm{R}^{\blacksquare} f(a)$$

and $y \,\mathrm{R}\, z$ so $z \in f(a)$ i.e. $a \in f^*(z)$.

(16) If $\sigma\langle f^*(y)\rangle \in f^\natural(x)$ and $\Diamond a \in f^*(y)$ then

$$y \in f(\Diamond a) \subseteq \Diamond^{\mathrm{R}} f(a) \subseteq \mathrm{R}^{\blacklozenge} f(a)$$

so there exists $z \in X$ such that $y \,\mathrm{R}\, z$ and $z \in f(a)$. But then $\sigma\langle f^*(y), f^*(z)\rangle \in f^\natural(x)$ and $a \in f^*(z)$.

Next we claim that $(f^\sharp)^{\leftarrow}\phi_A = f$. For

$$\begin{aligned} x \in (f^\sharp)^{\leftarrow}\phi_A(a) &\iff f^\sharp(x) \in \phi_A(a) \\ &\iff a \in f^*(x) \\ &\iff x \in f(a) \end{aligned}$$

It follows that $f^\sharp$ is continuous, since $(f^\sharp)^{\leftarrow}\phi_A(a) = f(a) \in \mathfrak{O}$. To show that $f^\sharp$ is a continuous relational morphism, we still need to check conditions (11) and (12):

(11) If $x \,\mathrm{R}\, y$ then $\langle f^*(x), f^*(y)\rangle \in f^\natural(x)$ and $f^\natural(x)(f^*(y)) = f^\natural(y)$, so $f^\sharp(x) \,\mathrm{R}_A\, f^\sharp(y)$.

(12) If $f^\sharp(x) \,\mathrm{R}_A [u]$ and $[u] \in \phi_A(a)$ then there is a tree t satisfying $f^\natural(x) \sim t$ and $\langle f^*(x), \mathrm{rt}(u)\rangle \in t$; also $a \in \mathrm{rt}(u)$. By the definition of $\sim$, there is a completely prime filter $q \ni a$ satisfying $\langle f^*(x), q\rangle \in f^\natural(x)$. Then $q = f^*(y)$ for some y such that $x \,\mathrm{R}\, y$; since $a \in q$, $f^\sharp(y) \in \phi_A(a)$.

So the existence of $f^\sharp$ is established.

Next we prove uniqueness. Let

$$g : (X, \mathfrak{O}, \mathrm{R}) \to (\mathrm{tp}(A), \mathfrak{A}, \mathrm{R}_A)$$

be a continuous relational morphism satisfying $(g)^{\leftarrow}\phi_A = f$, and choose $g^0 : X \to \mathrm{tree}(A)$ such that for all $x \in X$, $[g^0(x)] = g(x)$. We will show that $g^0(x) \sim f^\natural(x)$.

Note first that

$$\begin{aligned} a \in f^*(x) \iff x \in f(a) \iff x \in (g)^{\leftarrow}\phi_A(a) \iff g(x) \in \phi_A(a) \\ \iff a \in \mathrm{rt}(g^0(x)) \end{aligned}$$

so

$$\mathrm{rt}(g^0(x)) = f^*(x) = \mathrm{rt}(f^\natural(x)) \tag{17}$$

If $\langle f^*(x), f^*(y)\rangle \in f^\natural(x)$ and $a \in f^*(y)$ then $x \,\mathrm{R}\, y$ so $g(x) \,\mathrm{R}_A\, g(y)$ by condition (11), i.e. there is a tree s satisfying $s \sim g^0(x)$, $\langle \mathrm{rt}(s), \mathrm{rt}(g^0(y))\rangle \in s$ and $s(\mathrm{rt}(g^0(y))) \sim g^0(y)$. By (17), $\langle f^*(x), f^*(y)\rangle \in s$, so since $s \sim g^0(x)$ there is a $q \ni a$ such that $\langle f^*(x), q\rangle \in g^0(x)$.

If $\langle f^*(x), p\rangle \in g^0(x)$ and $a \in p$ then $g^0(x)(p) \in \mathrm{tree}(A)$ by lemma 5; so $g(x) \,\mathrm{R}_A [g^0(x)(p)]$ and $[g^0(x)(p)] \in \phi_A(a)$. By condition (12), there is a $y \in X$ such that $x \,\mathrm{R}\, y$ and $g(y) \in \phi_A(a)$, so $\langle f^*(x), f^*(y)\rangle \in f^\natural(x)$ and $a \in f^*(y)$.

Therefore $g(x) = [g^0(x)] = [f^\natural(x)] = f^\sharp(x)$, and uniqueness is established. $\Box$

It is a general result of category theory that any adjunction restricts to an equivalence between the full subcategories of objects where the unit or counit is an isomorphism. In the case of a contravariant adjunction this gives rise to a duality, assuming there are any such objects.

For any topological relational structure $(X, \mathfrak{O}, \mathrm{R})$, let $\psi_X = (1_{\mathfrak{O}})^\sharp$ be the unit on the topological side of the adjunction.

Corollary 9. The adjunction of theorem 2 restricts to a duality between those pre-modal frames which are isomorphic to one in the image of $\mathcal{O}$, and those topological relational structures which are isomorphic to one in the image of $\mathcal{G}$.

Proof. We only have to show that $\phi_{\mathcal{O}(X)}$ and $\psi_{\mathcal{G}(A)}$ are epi, since they are necessarily split monic. But ϕ_A is always onto, by definition of the topology on $\mathrm{tp}(A)$.

Let $t \in \mathrm{tree}(\mathcal{OG}(A))$. We will construct a tree $t_* \in \mathrm{tree}(A)$ such that $\psi_{\mathcal{G}(A)}(t_*) = [t]$ to show that $\psi_{\mathcal{G}(A)}$ is onto. First, if p is a completely prime filter in $\mathcal{OG}(A)$, let

$$p_* = \{a \in A | \phi_A(a) \in p\}$$

which is a completely prime filter in A. Next, define

$$t_* = \{\langle (p_1)_*, \ldots, (p_n)_* \rangle | \langle p_1, \ldots, p_n \rangle \in t\}$$

Conditions (13) and (14) are immediate; for the others:

(15) If $\sigma_* \langle p_*, q_* \rangle \in t_*$ and $\Box a \in p_*$ then $\sigma\langle p, q\rangle \in t$ and $\phi_A(\Box a) \in p$. But $\phi_A(\Box_a) \subseteq \Box^{\mathrm{R}_A} \phi_A(a)$ so $\Box^{\mathrm{R}_A} \phi_A(a) \in p$ and because t is a tree, $\phi_A(a) \in q$ i.e. $a \in q_*$.

(16) If $\sigma_* \langle p_* \rangle \in t_*$ and $\Diamond a \in p_*$ then $\sigma\langle p \rangle \in t$ and $\phi_A(\Diamond a) \in p$. But $\phi_A(\Diamond a) \subseteq \Diamond^{\mathrm{R}_A} \phi_A(a)$ so $\Diamond^{\mathrm{R}_A} \phi_A(a) \in p$ and because t is a tree, there is a $q \ni \phi_A(a)$ such that $\sigma\langle p, q \rangle \in t$. Then $\sigma_* \langle p_*, q_* \rangle \in t_*$ and $a \in q_*$.

Unravelling the construction of theorem 2 shows that $\psi_{\mathcal{G}(A)}(t_*) = [s]$ where

$$s = \{\langle [t_1]^*, \ldots, [t_n]^* \rangle | t_1 \sim t_* \ \& \ [t_j] \, \mathrm{R}_A [t_{j+1}]\}$$
$$[t]^* = \{\phi_A(a) | a \in \mathrm{rt}(t)\}$$

So $\mathrm{rt}(s) = \{\phi_A(a) | a \in \mathrm{rt}(t_*)\} = \{\phi_A(a) | \phi_A(a) \in \mathrm{rt}(t)\} = \mathrm{rt}(t)$.

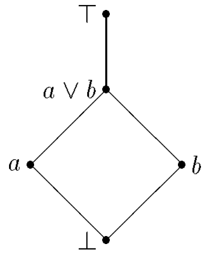
If $\langle [t_1]^*, [t_2]^* \rangle \in s$ and $\phi_A(a) \in [t_2]^*$ then $[t_*] \, \mathrm{R}_A [t_2]$ and $a \in \mathrm{rt}(t_2)$, so there is a tree t' such that $t_* \sim t'$ and $\langle \mathrm{rt}(t_*), \mathrm{rt}(t_2) \rangle \in t'$. By the definition of $\sim$, there is a completely prime filter p such that $\langle \mathrm{rt}(t_*), p_* \rangle \in t_*$ and $a \in p_*$; which means that $\langle \mathrm{rt}(t), p \rangle \in t$ and $a \in p$.

If $\langle \mathrm{rt}(t), p \rangle \in t$ and $\phi_A(a) \in p$ then $\langle \mathrm{rt}(t)_*, p_* \rangle \in t_*$ and $a \in p_*$, so $t_*(p_*) \in \phi_A(a)$, which means that $\langle \mathrm{rt}(t), [t_*(p_*)]^* \rangle \in s$ and $\phi_A(a) \in [t_*(p_*)]^*$.

Therefore $t \sim s$, and we have shown that $\psi_{\mathcal{G}(A)}$ is onto. $\Box$

In the non-modal case, topological spaces which are homeomorphic to the space of points of an ideal are sober, which implies T_0. We now present an example to show that this separation axiom fails in the modal case.

Let A be the frame



with the modality

$$\Box(x) = x$$
$$\Diamond(a \vee b) = \top$$
$$\Diamond(x) = x \quad (\text{if } x \neq a \vee b)$$

This frame has three completely prime filters

$$\begin{aligned} p_\top &= \{\top\} \\ p_a &= \{a, a \vee b, \top\} \\ p_b &= \{b, a \vee b, \top\} \end{aligned}$$

and five equivalence classes of trees, represented by

$$\begin{aligned} t_a &= p_a\text{—}p_a\text{—}p_a\text{—}\cdots \\ t_b &= p_b\text{—}p_b\text{—}p_b\text{—}\cdots \\ s_a &= p_\top\text{—}p_a\text{—}p_a\text{—}\cdots \\ s_b &= p_\top\text{—}p_b\text{—}p_b\text{—}\cdots \\ s_{a\vee b} &= p_\top \langle \begin{matrix} p_a\text{—}p_a\text{—}\cdots \\ p_b\text{—}p_b\text{—}\cdots \end{matrix} \end{aligned}$$

Since they have the same root, s_a, s_b and $s_{a\vee b}$ are not separated, so $\mathcal{G}(A)$ is not T_0.

5 Modally Spectral Frames

Theorem 1 shows how to construct a universal pre-modal frame on any pre-modal distributive lattice, and theorem 2 shows how to construct a universal topological relational structure from the resulting pre-modal frame. In two steps, therefore, we can construct a universal model. In this section we analyse precisely which topological relational structures arise in this way, and as a corollary, prove completeness for the axioms (1–4).

Recall that an element a of a frame $(A, \leqslant)$ is **compact** (or **finite**) iff

- if $S \subseteq A$ satisfies $\bigvee S \geqslant a$ then there exists a finite set $F \subseteq S$ such that $\bigvee F \geqslant a$.

We write the set of compact elements of A as $K(A)$. Recall that the frame A is **spectral** (or **coherent**) iff

- finite meets of compact elements are compact, i.e. $\top \in K(A)$ and if $a, b \in K(A)$ then $a \wedge b \in K(A)$, and
- every element is expressible as a join of compact elements, in fact for all $a \in A$

$$a = \bigvee \{b \in K(A) | b \leqslant a\} \tag{18}$$

Recall also that a frame is spectral if, and only if, it is isomorphic to the frame of ideals of a distributive lattice. The isomorphism is given by

$$\begin{aligned} \theta_A &: A \simeq \mathrm{Idl}(K(A)) \\ \theta_A(a) &= \{b \in K(A) | b \leqslant a\} \end{aligned} \tag{19}$$

Finally, recall that there is a bijection between completely prime filters p of A and prime filters f of $K(A)$ given by

$$a \in p \iff \exists b \in f.\ b \leqslant a \tag{20}$$

The aim of this section is to generalise these results to the modal case.

Definition. A pre-modal frame $(A, \leqslant, \Box, \Diamond)$ is **modally spectral** if

- the underlying frame $(A, \leqslant)$ is spectral, and

- the modal operations $\Box$ and $\Diamond$ preserve directed joins, and map compact elements to compact elements.

Proposition 10. A pre-modal frame is modally spectral if, and only if, it is isomorphic to the pre-modal frame of ideals (defined as in proposition 1) of a pre-modal distributive lattice.

Proof. If $(B, \leqslant, \Box, \Diamond)$ is a pre-modal distributive lattice, then we know from the non-modal case that $(\mathrm{Idl}(B), \subseteq)$ is spectral, and that $I \in \mathrm{Idl}(B)$ is compact iff it is principal. But

$$\begin{aligned}\Box(\downarrow b) &= \{a | \exists c \leqslant b.\ a \leqslant \Box c\} \\ &= \downarrow \Box b\end{aligned}$$

so $\Box$ preserves compactness, and $\Diamond$ similarly.

If $D \subseteq \mathrm{Idl}(B)$ is directed, then $\bigvee D = \bigcup D$ so

$$\begin{aligned}\Box(\bigvee D) = \Box(\bigcup D) &= \{a | \exists I \in D.\ \exists b \in I.\ a \leqslant \Box b\} \\ &= \bigcup_{I \in D} \{a | \exists b \in I.\ a \leqslant \Box b\} \\ &= \bigvee_{I \in D} \Box I\end{aligned}$$

so $\Box$ preserves directed joins, and $\Diamond$ similarly. Therefore $(\mathrm{Idl}(B), \subseteq, \Box, \Diamond)$ is modally spectral.

Conversely, if $(A, \leqslant, \Box, \Diamond)$ is modally spectral, then we know from the non-modal case that $(K(A), \leqslant)$ is a distributive lattice, and that (19) defines an isomorphism of frames. Since $\Box$ and $\Diamond$ are assumed to restrict to $K(A)$, it remains only to show that the defined operations on $\mathrm{Idl}(K(A))$ agree with the originals. But every element of A is expressible as a directed join of compact elements by (18), so

$$\begin{aligned}\Box a &= \Box \bigvee \{k \in K(A) | k \leqslant a\} \\ &= \bigvee \{\Box k | k \in K(A) \ \&\ k \leqslant a\}\end{aligned}$$

but this set is directed, which means for compact l that $l \leqslant \Box a$ iff there is $k \in K(A)$ such that $k \leqslant a$ and $l \leqslant \Box k$. Therefore

$$\begin{aligned}\theta_A(\Box a) &= \{l \in K(A) | l \leqslant \Box a\} \\ &= \{l \in K(A) | \exists k \in K(A).\ k \leqslant a \ \&\ l \leqslant \Box k\} \\ &= \Box\{k \in K(A) | k \leqslant a\} \\ &= \Box \theta_A(a)\end{aligned}$$

Similarly, $\theta_A(\Diamond a) = \Diamond \theta_A(a)$, so θ_A is an isomorphism of pre-modal frames. $\Box$

When studying the trees of a modally spectral frame A, we need to know how they relate to the distributive lattice $K(A)$. This is straightforward once we define the trees of $K(A)$.

Definition. **A tree** of a pre-modal distributive lattice B is a nonempty set of nonempty finite sequences of prime filters of B, satisfying conditions (13–16).

The set of trees of B is denoted $\mathrm{tree}(B)$.

It will always be clear from the context whether we mean trees in this sense or trees in the old sense.

Lemma 11. Let $(B, \leqslant, \Box, \Diamond)$ be a pre-modal distributive lattice. The trees of B (as defined above) are in bijective correspondence with the trees of $\mathrm{Idl}(B)$ (considered as a pre-modal frame).

Proof. The bijection is simply the extension of (20) to trees. $\Box$

In proposition 3 we showed that any pre-modal frame in the scope of the duality of corollary 9 satisfies axioms (1–4), which can be interpreted as a soundness result. The main logical result of this section is the corresponding completeness result: that any modally spectral frame which satisfies axioms (1–4) is in the scope of the duality, i.e. that ϕ_A is a modal isomorphism. The following lemma is useful in proving isomorphisms.

Lemma 12. A modal frame morphism is iso if, and only if, it is one-one, onto and strict.

Proof. A frame morphism f is iso iff it is 1-1 and onto, so we have only to prove that the inverse f^{-1} is a modal frame morphism iff f is strict. But

$$\begin{aligned} \forall a.\ f^{-1}(\Box a) \leqslant \Box f^{-1}(a) &\iff \forall a.\ \Box a \leqslant f(\Box f^{-1}(a)) \\ &\iff \forall a.\ \Box f(a) \leqslant f(\Box a) \end{aligned}$$

and similarly for $\Diamond$. □

Next we prove a generalisation of the prime ideal theorem to the modal case: this shows that a distributive lattice has “enough” trees.

Lemma 13. Let $(B, \leqslant, \Box, \Diamond)$ be a pre-modal distributive lattice satisfying axioms (1–4). If I is an ideal of B and F is a filter of B such that $I \cap F = \emptyset$, then there exists a tree t of B such that $F \subseteq \mathrm{rt}(t)$ and $I \cap \mathrm{rt}(t) = \emptyset$.

Proof. Define t_j inductively as follows.

By the prime ideal theorem, there exists a prime filter f s.t. $F \subseteq f$ and $I \cap f = \emptyset$; let $t_1 = \{\langle f\rangle\}$.

If $\sigma\langle g\rangle \in t_j$ and $\Diamond c \in g$ then

$$\uparrow (c \wedge \Box_g) = \{d | \exists e.\ \Box e \in g \ \&\ c \wedge e \leqslant d\}$$

is a proper filter since

- If $c \wedge e \leqslant d$ and $c \wedge e' \leqslant d'$ then $c \wedge (e \wedge e') \leqslant d \wedge d'$ but $\Box(e \wedge e') \geqslant \Box e \wedge \Box e' \in g$ by (2), so it is closed under $\wedge$.
- $c \wedge \top \leqslant \top$ but $\Box\top \geqslant \top \in g$ by (1), so it contains $\top$.
- If $c \wedge e \leqslant \bot$ and $\Box e \in g$ then $\Diamond c \wedge \Box e \in g$ but $\Diamond c \wedge \Box e \leqslant \Diamond(c \wedge e) \leqslant \Diamond\bot \leqslant \bot$ by (3) and (4), contradicting $\bot \notin g$.

So by the prime ideal theorem, we can choose a prime filter $h(c)$ containing $\uparrow (c \wedge \Box_g)$. Let

$$t_{j+1} = \{\sigma\langle g, h(c)\rangle | \sigma\langle g\rangle \in t_j \ \&\ \Diamond c \in g\}$$

We claim that $t = \bigcup_j t_j$ is a tree of B. The conditions (13) and (14) are trivial, and:

(15) If $\sigma\langle g, h(c)\rangle \in t_{j+1}$ and $\Box e \in g$ then $e \in \uparrow (c \wedge \Box_g) \subseteq h(c)$.

(16) If $\sigma\langle g\rangle \in t_j$ and $\Diamond c \in g$ then $c \in \uparrow (c \wedge \Box_g) \subseteq h(c)$ and $\sigma\langle g, h(c)\rangle \in t$.

Then t is as required because $\mathrm{rt}(t) = f$. □

This allows us to prove the hypotheses of lemma 12.

Lemma 14. If $(A, \leqslant, \Box, \Diamond)$ is a modally spectral frame satisfying axioms (1–4), then ϕ_A is one-one.

Proof. Without loss of generality, we take $A = \mathrm{Idl}(B)$ for some modal distributive lattice B, which itself satisfies axioms (1–4). We will show that ϕ_A is an order embedding.

Let $I, J \in \mathrm{Idl}(B)$ such that $I \nsubseteq J$, choose $a \in I$ such that $a \notin J$, and apply lemma 13 to J and $\uparrow a$ to get a tree t of B.

Let t' be the corresponding tree of $\mathrm{Idl}(B)$; since $a \in I \cap \mathrm{rt}(t)$ we have $[t'] \in \phi_A(I)$, and since $J \cap \mathrm{rt}(t) = \emptyset$ we have $[t'] \notin \phi_A(J)$. So $\phi_A(I) \nsubseteq \phi_A(J)$. □

Lemma 15. If $(A, \leqslant, \Box, \Diamond)$ is a modally spectral frame satisfying axioms (1–4), then for all $a \in A$, $\Box^{\mathrm{R}_A} \phi_A(a) \subseteq \phi_A(\Box a)$.

Proof. As before we take $A = \mathrm{Idl}(B)$.

Let $I, J \in \mathrm{Idl}(B)$ satisfy $\phi_A(I) \subseteq \Box^{\mathrm{R}_A} \phi_A(J)$, and let $b \in I \setminus \Box J$. By lemma 13 there is a tree t s.t. $b \in \mathrm{rt}(t)$ and $\Box J \cap \mathrm{rt}(t) = \emptyset$. Now

$$F = \{c \in B | \Box c \in \mathrm{rt}(t)\}$$

is a filter disjoint from J, since:

- If $c \leqslant d$ and $\Box c \in \mathrm{rt}(t)$ then $\Box c \leqslant \Box d$ so $\Box d \in \mathrm{rt}(t)$.
- If $\Box c \in \mathrm{rt}(t)$ and $\Box d \in \mathrm{rt}(t)$ then $\Box(c \wedge d) \geqslant \Box c \wedge \Box d \in \mathrm{rt}(t)$.
- $\Box \top \geqslant \top \in \mathrm{rt}(t)$
- If $a \in F \cap J$ then $\Box a \in \mathrm{rt}(t) \cap \Box J$ which contradicts the choice of t.

By lemma 13 (again) there is a tree s satisfying $J \cap \mathrm{rt}(s) = \emptyset$ and $F \subseteq \mathrm{rt}(s)$. Let

$$u = t \cup \{\langle \mathrm{rt}(t) \rangle \sigma | \sigma \in s\}$$

It is easy to see that u is a tree, and since $b \in \mathrm{rt}(u)$, the tree $u' \in \mathrm{tree}(A)$ corresponding to u satisfies $[u'] \in \phi_A(I)$. Let $s' \in \mathrm{tree}(A)$ be the tree corresponding to s; by construction, $[u'] \, \mathrm{R}_A [s']$, but $[s'] \notin \phi_A(J)$ contradicting $\phi_A(I) \subseteq \Box^{\mathrm{R}_A} \phi_A(J)$.

So $I \subseteq \Box J$, and therefore $\Box^{\mathrm{R}_A} \phi_A(J) \subseteq \phi_A(\Box J)$. $\Box$

Lemma 16. If $(A, \leqslant, \Box, \Diamond)$ is a modally spectral frame satisfying axioms (1–4), then for all $a \in A$, $\Diamond^{\mathrm{R}_A} \phi_A(a) \subseteq \phi_A(\Diamond a)$.

Proof. As before we take $A = \mathrm{Idl}(B)$.

Let $I, J \in \mathrm{Idl}(B)$ satisfy $\phi_A(I) \subseteq \Diamond^{\mathrm{R}_A} \phi_A(J)$, and let $b \in I \setminus \Diamond J$. By the prime ideal theorem, there is a prime filter f of B such that $b \in f$ and $f \cap \Diamond J = \emptyset$.

For each $\Diamond c \in f$, the set

$$\uparrow (c \wedge \Box_f) = \{d | \exists e. \, \Box e \in f \; \& \; c \wedge e \leqslant d\}$$

is a filter disjoint from J since

- If $c \wedge e \leqslant d$ and $c \wedge e' \leqslant d'$ then $c \wedge (e \wedge e') \leqslant d \wedge d'$ but $\Box(e \wedge e') \geqslant \Box e \wedge \Box e' \in f$ by (2), so it is closed under $\wedge$.
- $c \wedge \top \leqslant \top$ but $\Box \top \geqslant \top \in f$ by (1), so it contains $\top$.
- If $c \wedge e \leqslant d \in J$ and $\Box e \in f$ then $\Diamond c \wedge \Box e \in f$ but $\Diamond c \wedge \Box e \leqslant \Diamond(c \wedge e) \leqslant \Diamond d \in \Diamond J$ by (3) and (4), contradicting $\Diamond J \cap f = \emptyset$.

Apply lemma 13 to get, for each such c, a tree $t(c)$ such that $\uparrow (c \wedge \Box_f) \subseteq \mathrm{rt}(t(c))$ and $J \cap \mathrm{rt}(t(c)) = \emptyset$. Let

$$u = \{\langle f \rangle \sigma | \exists c. \, \Diamond c \in f \; \& \; \sigma \in t(c)\}$$

It is easy to see that u is a tree, and since $b \in \mathrm{rt}(u)$, the tree $u' \in \mathrm{tree}(A)$ corresponding to u satisfies $[u'] \in \phi_A(I)$.

Since $\phi_A(I) \subseteq \Diamond^{\mathrm{R}_A} \phi_A(J)$, there is $s' \in \mathrm{tree}(A)$ such that $[u'] \, \mathrm{R}_A [s']$ and $[s'] \in \phi_A(J)$; let $a \in \mathrm{rt}(s') \cap J$. By construction of u, there is no prime filter $g \ni a$ such that $\langle f, g \rangle \in u$, contradicting the definition of R_A.

So $I \subseteq \Diamond J$, and therefore $\Diamond^{\mathrm{R}_A} \phi_A(J) \subseteq \phi_A(\Diamond J)$. $\Box$

Theorem 3. If $(A, \leqslant, \Box, \Diamond)$ is a modally spectral frame satisfying axioms (1–4), then it is in the scope of the duality of corollary 9.

Proof. Put together lemmas 12 to 16 with the fact that ϕ_A is always onto (by definition of the topology on tree points) to show that ϕ_A is an isomorphism. $\Box$

Theorem 3 shows that axioms (1–4) are complete: if some other axiom holds in every topological relational structure, then it certainly holds in the space of tree points of the frame of ideals of the Lindenbaum algebra of the propositional language with these axioms. But then, by theorem 3 it holds in the frame of ideals, and by lemma 2 it holds in the Lindenbaum algebra. Therefore it follows from the axioms.

It is now straightforward to find a duality theorem for modal distributive lattices. We define

Definition. A **modal distributive lattice** is a pre-modal distributive lattice satisfying axioms (1–4).

A **spectral relational structure** is a topological relational structure isomorphic to the space of tree points of a modally spectral frame.

A **spectral relational morphism** between spectral relational structures is a continuous relational morphism such that the inverse image of a compact open is compact.

Corollary 17. The category of modal distributive lattices and modal distributive-lattice morphisms is dual to the category of spectral relational structures and spectral relational morphisms.

6 Conclusions

The dualities described in this paper give a firm foundation to propositional intuitionistic modal logic. The semantics in terms of a set of worlds with a topology and an accessibility relation provide a natural and mathematically convincing generalisation of the Kripke semantics of classical modal logic, as well as a straightforward generalisation of the topological semantics of intuitionistic logic. Furthermore, the spectral case of the duality shows that the axioms (1–4) are complete, in just the same way as in the non-modal case.

Because of the similarity between the adjunction described in this paper and that between frames and topological spaces, it is tempting to ask how much further this work could be developed along the same lines. Several other dualities arise from the non-modal adjunction, including those between compact regular frames and compact Hausdorff spaces, and between Boolean algebras and Stone spaces; it seems probable that these could be generalised to the modal case. If the duality could be lifted from spaces to sheaves or even toposes, this could provide a semantics of predicate intuitionistic modal logic. The only major difference between the modal and non-modal adjunctions seems to be the lack, in the modal case, of a schizophrenic object.

There is less similarity between the structures described in this paper and those of classical modal logic. In the classical case, the worlds of a modal Boolean algebra are the maximal consistent sets: the same as used in the non-modal completeness theorem. The modality simply defines an accessibility relation on this set, and so we can generalise to the multi-modal case, or to relations with special properties, without changing the set of worlds. In the intuitionistic case, the points are trees, and (as can be seen from the example at the end of section 4) the set of tree points depends on the modality as well as on the order structure. In order to generalise to the multi-modal case it is clear that we need trees labelled by the index set (or signature) of the logic. Furthermore, if we intend one accessibility relation to be the inverse or transitive closure of another, these general labelled trees will not achieve this. Similarly, axioms (1–5,8) are shown to be sound with respect to relations which are continuous, but the trees defined in this paper will not give continuous relations without adaptation, so cannot be used to prove completeness.

In the author's opinion, these problems are not disadvantages of the techniques described in the paper, but fundamental problems of intuitionistic modal logic. The same kind of difficulties show up when we try to prove correspondence results for simple relational properties such as reflexivity, transitivity, and so on. Because the sets used in the classical proofs are not open, the same proofs do not go through. We must either restrict the class of models (demanding a new completeness result) or change the relational properties to take account of the topology. This is a fertile field for further work.

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