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Abstract

In this paper, we investigate a method of representing convex regions of the plane based on the *sinusoidal transform.* The sinusoidal transform is a member of the general class of geometrical representation schemes based on a 'boundary function'—a periodic function of one variable which picks out a bounded region of the plane in terms of the contour that encloses it. Various types of boundary function have been explored in the literature, but one feature they all share is that, being periodic, they can be approximated by a sequence of Fourier coefficients, which therefore serve to represent the original region.

The aim of the paper is to obtain a thorough understanding of the sinusoidal transform of its associated Fourier descriptors. We investigate the derivatives of the sinusoidal transform and establish an inverse transform. We obtain a characterization of the set of periodic functions of one variable which are the sinusoidal transforms of 'well-behaved' convex regions of the plane. We derive negative results concerning the prospects for extending the sinusoidal representation scheme to non-convex regions. Finally—and of more practical importance—we show how various geometrical quantities and relations—including overlap, inclusion, separation, distance, radius of curvature, perimeter length, area and congruence—can be extracted from the sinusoidal transforms of regions,and hence from their sinusoidal Fourier descriptors.

1 Background

The sinusoidal transform is a member of the general class of geometrical representation schemes based on 'boundary functions'. A boundary function is a periodic function of one variable which picks out a bounded region of the plane in terms of the contour that encloses it. Various types of boundary function have been explored in the literature, but one feature they all share is that, being periodic, they can be approximated by a sequence of Fourier coefficients, which therefore serve to represent the original region. For example, Zahn and Roskies [12] investigate the boundary function $\phi^*(s)$ which gives (a normalized version of) the tangent angle at any point of the contour as a function of the normalized arc-length to that point from a fixed starting point. Granlund [2], Persoon and Fu [6] and Lin and Hwang [4] consider instead the boundary functions $X(s)$ and $Y(s)$ giving the Cartesian coordinates of each boundary point any point of the contour as a function of the normalized arc-length to that point from a fixed starting point. And Kiryati and Maydan [3] analyse the boundary function $r(\phi)$ such that $(r(\phi), \phi)$ are the polar coordinates of the contour points relative to an origin within the region.

Exactly which geometric features of regions can be read off from their representations in terms of Fourier coefficients depends on the type of boundary function used. In this paper, we investigate what happens when we take the boundary function to be the *sinusoidal transform* (Pratt [7]). This method of representing convex regions is essentially equivalent to the 'rotating callipers' method (Toussaint [10]) and to the well-studied and widely-applied Hough- and Radon transforms (Leavers [5]). According to this method, a convex region of the plane can be represented by rotating it around the origin and plotting the height (the y-coordinate) of its highest and lowest points over time. Figure 1 shows some convex regions of the plane and their corresponding representations. Although sinusoidal transforms are limited to convex regions, they enjoy certain advantages over the boundary functions mentioned in the previous paragraph. In particular, regions do not have to be normalized for size prior to encoding, as they do for boundary functions based on arc-length; and regions are not required to contain the origin, as they are for the polar boundary functions. Spatial representation schemes based on similar geometrical ideas have also been proposed as models of spatial representation in animal brains (Touretzky, Redish and Wan [11]). However, our concern in this paper is not to apply, or even to recommend, the sinusoidal transform, but rather, to establish its properties.

The plan of the paper is as follows. We begin by introducing the sinusoidal transform, explaining its most basic properties. We then investigate the transform from a more analytical perspective, deriving further properties on our way. Armed with this analysis, we establish an inverse of the sinusoidal transform, and discuss the representation of convex regions based on sinusoidal Fourier descriptors.

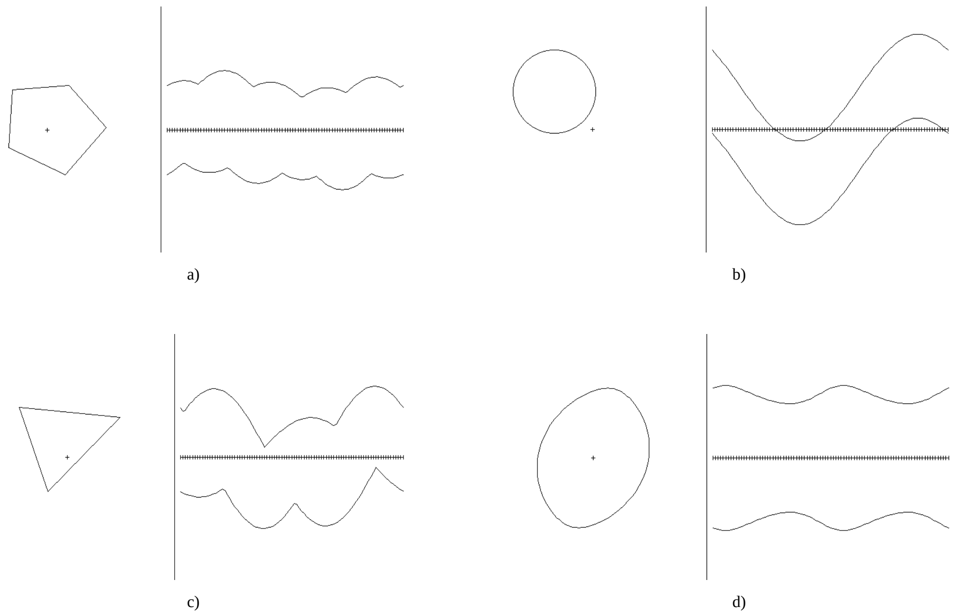


Figure 1: Some convex regions and their representations over one revolution

2 The sinusoidal transform

Given a vector $\vec{v}$ whose polar coordinates are (a, ϕ), we can associate with $\vec{v}$ the sine wave

$$w_{\vec{v}}(t) \quad = \quad a \sin(t + \phi) \tag{1}$$

with amplitude a and phase-lead ϕ. We shall say that $w_{\vec{v}}(t)$ is the sine wave *corresponding to* the vector $\vec{v}$. Geometrically, if we imagine the vector $\vec{v}$ rotating about the origin with constant angular velocity 1 radian per unit time, then the sine-wave corresponding to $\vec{v}$ is the graph obtained by plotting the height (i.e. the y-coordinate) of $\vec{v}$ against time. The following well-known fact states that we can add vectors by superposing their corresponding sine-waves:

Observation 1 *Let $\vec{v}_1$, $\vec{v}_2$ be vectors having polar coordinates (a_1, ϕ_1), (a_2, ϕ_2) respectively, and let $w_{\vec{v}_1}(t) = a_1 \sin(t + \phi_1)$, $w_{\vec{v}_2}(t) = a_2 \sin(t + \phi_2)$ be the corresponding sinusoidal waves of frequency 1. If $w(t)$ is the pointwise sum $w_{\vec{v}_1}(t) + w_{\vec{v}_2}(t)$, and if $\vec{v}$ is the vector sum $\vec{v}_1 + \vec{v}_2$, then $w(t)$ is also a sinusoidal wave of frequency 1, $w(t) = w_{\vec{v}}(t) = a \sin(t + \phi)$, where (a, ϕ) are the polar coordinates of the vector $\vec{v}$.*

We now show how a pair of periodic functions of one variable can represent a convex region of the plane.

Definition 1 *If S is a convex, compact set of vectors, we define the* sinusoidal hull corresponding to S *to be the pair of functions $\langle a_S(t), b_S(t) \rangle$:*

$$a_S(t) = \sup_{v \in S} w_{\vec{v}}(t) \qquad b_S(t) = \inf_{v \in S} w_{\vec{v}}(t).$$

(By the compactness of S, these limits will always exist.) In addition, we call the function $a_S(t)$ the sinusoidal transform *of S.*

Geometrically, if we imagine the region S rotating anti-clockwise about the origin with constant angular velocity of 1 radian per until time, then $a_S(t)$ is the graph obtained by plotting the height (i.e. the y-coordinate) of the highest point on S, and $b(t)$ is the graph obtained by plotting the height (i.e.

the y-coordinate) of the lowest point on S. That is, $a_S(t)$ and $b_S(t)$ give the projection of S on the y-axis as S is rotated about the origin (see figure 1). The ability to represent a convex region S of the plane using the functions $a_S(t)$ and $b_S(t)$ depends on the following observation (which is easy to verify from the geometrical interpretation of sinusoidal hulls):

Observation 2 *Let S be a compact, convex set of vectors, and let $\langle a_S(t), b_S(t)\rangle$ be the sinusoidal hull corresponding to S. If $\vec{v}$ is any vector, and $w_{\vec{v}}(t)$ its corresponding sine wave, then: (i) $\vec{v} \in S$ iff $\forall t,\ a_S(t) \geq w_{\vec{v}}(t)$; (ii) $\vec{v} \in S$ iff $\forall t,\ b_S(t) \leq w_{\vec{v}}(t)$.*

Thus, the function $a_S(t)$ (or, alternatively, $b_S(t)$) represents S uniquely, because it suffices to determine of any given vector $\vec{v}$, represented by a sine wave $w_{\vec{v}}(t)$, whether $\vec{v}$ lies in S. Although each of the functions $a_S(t)$ and $b_S(t)$ by itself contains all the information needed to represent S, we shall sometimes find it convenient to represent a region S using both functions—that is, using its sinusoidal hull.

If we represent convex regions by their sinusoidal hulls, a range of geometrical tests and operations becomes available. Suppose we want to translate a convex set S by a vector $\vec{v}$, where S is represented by the functions $\langle a_S(t),\ b_S(t)\rangle$ and $\vec{v}$ by the function $w_{\vec{v}}(t)$. The following theorem follows simply from observations 1 and 2:

Theorem 1 *Let S be a region represented by functions $\langle a_S(t), b_S(t)\rangle$, $\vec{v}$ a vector represented by the sine wave $w_{\vec{v}}(t)$ and θ a real number. Let $S_{\vec{v}}$ denote the region obtained by translating S by $\vec{v}$, and S_θ the region obtained by rotating S anti-clockwise about the origin through θ radians. Then $S_{\vec{v}}$ is represented by the functions $\langle a_{S_{\vec{v}}}(t), b_{S_{\vec{v}}}(t)\rangle$, where:*

$$a_{S_{\vec{v}}}(t) = a_S(t) + w_{\vec{v}}(t) \qquad b_{S_{\vec{v}}}(t) = b_S(t) + w_{\vec{v}}(t).$$

And S_θ is represented by the signals $\langle a_{S_\theta}(t), b_{S_\theta}(t)\rangle$, where:

$$a_{S_\theta}(t) = a_S(t+\theta) \qquad b_{S_\theta}(t) = b_S(t+\theta)$$

.

In the remainder of this section, we show how various relationships between a pair of convex regions can be read off from the sinusoidal hulls of those regions. Again, the theorems are so simple that we state them without proof.

Theorem 2 *If $\langle a_S(t), b_S(t)\rangle$ and $\langle a_T(t), b_T(t)\rangle$ are sinusoidal hulls corresponding to convex regions S and T, respectively, then*

1. *$S \cap T \neq \emptyset$ iff $\forall t\ a_T(t) \geq b_S(t)$*
2. *$S \subseteq T$ iff $\forall t\ a_T(t) \geq a_S(t)$.*

If S and T are non-overlapping convex regions, then we take the *separation* between them to be the length of the shortest line segment with endpoints in S and T. The following theorem is not trivial, but still straightforward. (Readers wishing to check it should note the observations in section 3 on the geometrical interpretation of the derivative of the sinusoidal transform.)

Theorem 3 *If $\langle a_S(t), b_S(t)\rangle$ and $\langle a_T(t), b_T(t)\rangle$ are sinusoidal hulls corresponding to compact, convex regions S and T, respectively, such that $S \cap T = \emptyset$, then the distance between S and T is given by*

$$d(S,T) \quad = \quad \sup(\{(b_T(t) - a_S(t)) | t \in [0, 2\pi)\})\ .$$

A more general measure of the distance between two compact convex regions—overlapping or not—is as follows. (See Eggleston [1] ch. 4 for a full analysis). Let d_1 be the smallest non-negative number such that every point in S is no more than distance d_1 from T, and let d_2 be the smallest non-negative number such that every point in T is no more than distance d_2 from S. Then we define $\Delta(S,T) = d_1 + d_2$.

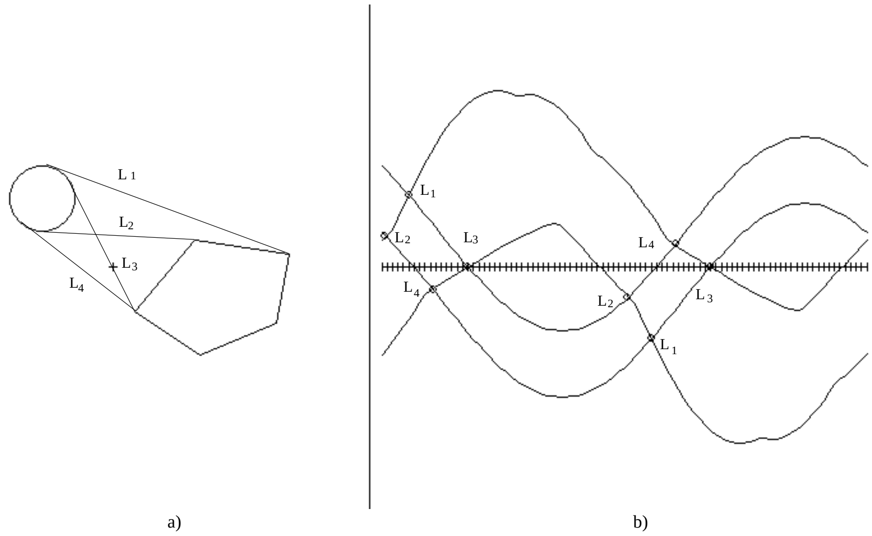


Figure 2: The eight crossing-points of two sinusoidal hulls

Theorem 4 *Let $a_S(t)$, $a_T(t)$ be the sinusoidal transforms of compact convex regions S and T, respectively. Then*

$$\Delta(S,T) \;=\; \sup(\{0\} \cup \{(a_S(t) - a_T(t)) | t \in [0, 2\pi)\}) + \sup(\{0\} \cup \{(a_T(t) - a_S(t)) | t \in [0, 2\pi)\})\ .$$

This result establishes the stability of sinusoidal representations, because 'similar' convex regions (according to the standard Δ-metric) yield 'similar' sinusoidal representations (according to an appropriate metric on bounded periodic functions).

Finally, we note in passing that, if S and T are non-overlapping convex regions, there exist four lines of support—and hence eight directed lines of support—between S and T, as shown in figure 2a). Notice that each directed line of support will have well-defined orientations (clockwise or anti-clockwise) about S and T. The sinusoidal hulls $\langle a_S(t), b_S(t)\rangle$ and $\langle a_T(t), b_T(t)\rangle$ will have eight crossing points as shown in figure 2b), corresponding to these eight directed lines of support. (Geometrically, if we imagine S and T rotating together about the origin, the crossing points occur at times when the lines of support are horizontal.) This correspondence is the basis for the use of sinusoidal hulls for route-planning, as described in Pratt [9].

3 Analysing the sinusoidal transform

In this section, we analyse the sinusoidal transform in more detail and lay the groundwork for section 4, where we investigate its inverse. Along the way, we also establish some surprising properties of the sinusoidal transform.

Let S be a compact, convex region of the plane. Each point on the boundary of S lies on one or more tangents to S. (We take a *tangent* to be a line intersecting the boundary of S but not its interior). Some tangents to S may lie on a straight-line segment of S's boundary; all other tangents to S will intersect S in just one point. Likewise, some points on the boundary of S will may on more than one tangent (we refer to these points as *kinks* in the boundary of S); all other points will lie on just one tangent. If we give the boundary of S an orientation—say, clockwise—any tangent to S picks up a direction in the obvious way, and can be uniquely identified by the angle t it makes with the horizontal (measured clockwise from the positive x-axis) in the range $[0, 2\pi)$. Thus, every angle t in the range $[0, 2\pi)$ corresponds to either a unique point $(x_S(t),\ y_S(t))$ on the boundary of S, or to a straight line segment of the boundary of S.

It follows that, if the boundary of S contains no straight line segments, it can be described in parametric form by the pair of functions: $C_S(t) = (x_S(t), y_S(t))$. Geometrically: C_S tours the boundary of S clockwise in such a way that, at any time t, we are at a point where there is a tangent to S making angle t with the x-axis. Kinks in the boundary of S correspond to intervals over which $C_S(t)$ is constant. If the boundary of S contains a straight line segment corresponding to the tangent angle t, then t does not pick out a unique boundary point on S. However, the left and right limits, $x_S(t^-)$, $y_S(t^-)$, $x_S(t^+)$ and $y_S(t^+)$ must exist. Therefore, we can still represent S in parametric form $C_S(t)$ by allowing x_S and y_S to be discontinuous at t, under the convention that the gap is to be 'filled in' by the straight line segment from $C_S(t^-)$ to $C_S(t^+)$. In order to make C_S total, we set $(x_S(t), y_S(t)) = (x_S(t^-), y_S(t^-))$—corresponding to the *first* point encountered on that line segment (proceeding clockwise around the boundary of S).

It will simplify the analysis if we restrict our attention to convex regions having well-behaved boundaries in the following sense.

Definition 2 *Let S be a compact, convex region of the plane. We say that S is* well-behaved *if S has interior points and the boundary of S has positive, finite radius of curvature at any point, except possibly for a finite number of straight line segments and a finite number of kinks.*

Concentrating on well-behaved regions simplifies the analysis without ruling out any regions of practical interest.

We begin by establishing the significance of the first derivative $a'_S(t)$. Figure 3 shows S together with a boundary point P, such that S's boundary in the region of P contains no straight line segments. As we can see, the sine wave $w(t)$ corresponding to P touches $a_S(t)$ internally at the time t_0 when P is uppermost on S. Thus, not only do we have $a_S(t_0) = w(t_0)$ but also $a'_S(t_0) = w'(t_0)$. But $w'(t)$ simply gives the x-coordinate of P as P is rotated about the origin along with S. Thus, just as $a_S(t)$ gives the y-coordinate of the highest point on S as S is rotated about the origin, so $a'_S(t)$ gives the x-coordinate of that same point. A rigorous proof of the following theorem can be given using a limit-argument, but the details are routine.

Theorem 5 *Let S be a well-behaved, compact, convex region of the plane, and suppose S to rotate anticlockwise about the origin with unit angular verlocity. Let P be a boundary point of S, and suppose P to rotate with S.*

If the boundary of S contains no straight line segments in the region of P, and t_0 is a time when P is uppermost, then $a'_S(t_0)$ exists, and $(a'_S(t_0), a_S(t_0))$ are the coordinates of this point at that time.

If P lies at the start of a straight line segment of the boundary of S (proceeding clockwise around S), and that line segment is uppermost at time t_0, then $a'_S(t_0^-)$ exists, and $(a'_S(t_0^-), a_S(t_0))$ are the coordinates of P at that time.

If P lies at the end of a straight line segment of the boundary of S (proceeding clockwise around S), and that line segment is uppermost at time t_0, then $a'_S(t_0^+)$ exists, and $(a'_S(t_0^+), a_S(t_0))$ are the coordinates of P at that time.

As an immediate corollary of theorem 5, if S's boundary contains a straight-line segment, which becomes uppermost on S at time t_0, then $a'_S(t)$ is discontinuous at t_0. (Actually, it is easy to show that $a'_S(t_0)$ does not exist.) Moreover, the length l_t of this line segment is given by the size of the discontinuity:

$$l_t = a'_S(t_0^+) - a_S(t_0^-)\ . \tag{2}$$

Note that this quantity must, therefore, be positive.

Finally, having dealt with straight-line segments in S's boundary, what can we say of kinks? If P lies at a kink in S's boundary, then, assuming that S has some interior points, there exist t_1, t_2 with $0 < t'' - t' < \pi$, such that a tangent to S passes through P just in case the angle of that tangent lies in the interval $[t', t'']$ (modulo 2π). The quantity $t'' - t'$ is the external angle of the kink. Clearly, $a_S(t)$ will be a sine wave over the interval $[t', t'']$.

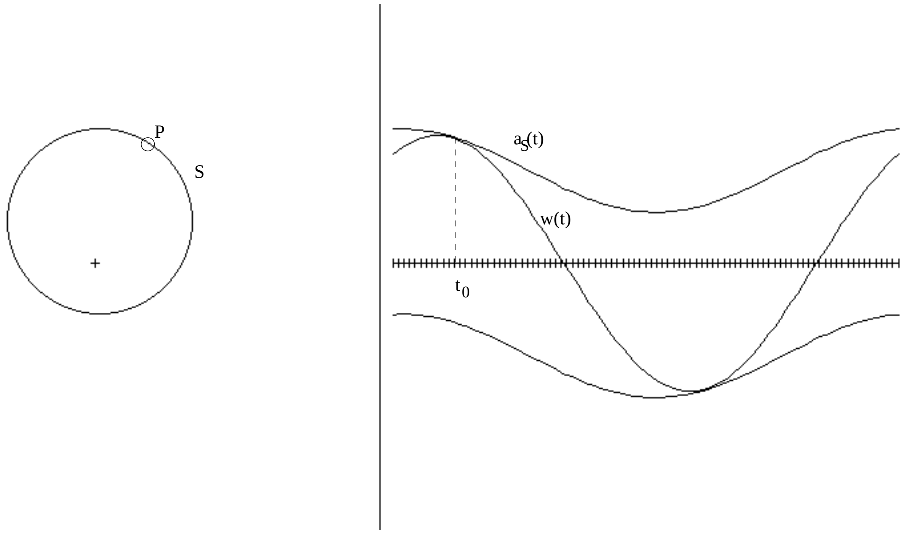


Figure 3: A boundary point to a region and its corresponding sine wave.

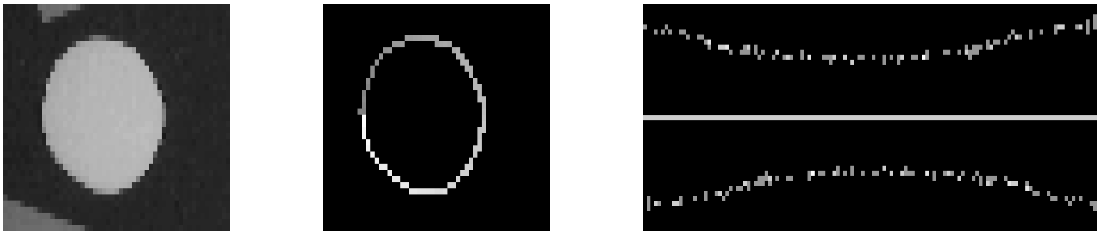
To summarize: (i) a point P on a smoothly curving portion of S's boundary corresponds to a single time (in each revolution) lying within some non-sinusoidal region of $a_S(t)$; (ii) all points P on a straight-line segment of S's boundary correspond to a single time (in each revolution) at which the derivative $a'_S(t)$ is discontinuous, with the length of the line segment equal to the size of the discontinuity; (iii) a point P at a kink on S's boundary corresponds to all points within a time interval (in each revolution) over which $a_S(t)$ is a pure sine wave, with the external angle of the kink equal to the duration of the interval.

Having established the geometrical interpretation of $a'_S(t)$, we can now relate $a_S(t)$ to the curve $C_S(t)$. Let S be a well-behaved, compact, convex region of the plane. By simple rotation of axes, we have:

$$a_S(t) = x_S(t)\sin t + y_S(t)\cos t \tag{3}$$
$$a'_S(t) = x_S(t)\cos t - y_S(t)\sin t\ . \tag{4}$$

The first of these equations holds for all t, the second, for all t such that $C_S(t)$ is not on a straight line segment of S's boundary.

Equation (3) tells us that the sinusoidal transform is less remote from practical applications of spatial reasoning than we may have thought, bearing in mind that spatial information on which spatial reasoning operates will often be visually acquired. In practice, computer processing of images proceeds by first detecting boundaries of regions in a way that automatically delivers, for each boundary-point detected, the slope of the tangent to the boundary at that point. That is, in terms of our parametric equations, boundary points $(x_S(t), y_S(t))$ are naturally detected along with their tangent slope t. This means that equation (3) can be applied to generate sinusoidal transforms from images. Figure 4a) shows a small region (excised from a photograph); figure 4b) shows the result of applying standard filters to calculate the tangent angle at all boundary points (with the shade of grey representing the tangent angle); figure 4c) shows the (approximation to) the sinusoidal transform obtained by applying equation (3) to the points in figure 4b). Many approaches to image processing use representations which are, in effect, sinusoidal hulls or variants thereof. In particular, there is clearly a close relationship between the sinusoidal transform and the Hough transform (see, for example, Leavers [5]). However, in this paper, we shall not concern ourselves with the practical difficulties of extracting the sinusoidal transform (or its relatives) from an image.



a) b) c)

Figure 4: The generation of a sinusoidal transform from an image.

Let us now turn our attention to the significance of the second derivative of the sinusoidal transform, $a''_S(t)$. Again, let S be a well-behaved, compact, convex region of the plane. Suppose that $C_S(t_0)$ is a point on the boundary of S lying neither on a straight-line segment nor at a kink. By hypothesis, S's boundary has a well-defined, positive radius of curvature at $C_S(t_0)$, which we may denote $r_S(t_0)$. We shall now show:

$$r_S(t_0) \quad = \quad a_S(t_0) + a''_S(t_0) \ . \tag{5}$$

We suppose first that $C_S(t_0)$ is the origin, O. (This simplifies the mathematics.) Figure 5 shows a nearby point P with coordinates $C_S(t_0 + \delta t)$, together with normals to the boundary of S subtended from O and P. (These must exist, by the fact that the boundary of S is smoothly curving in the region of O.) Let C be the point of intersection of these normals. We may take the radius of curvature to be the limit of the length $r_{\delta t}$ of OC as $\delta t \to 0$, where this limit exists. By theorem 5, $a'_S(t)$ exists in some region of t_0. Moreover, $a'_S(t_0) = 0$ and $a'_S(t_0 + \delta t) = l_{\delta t}$, where $l_{\delta t}$ is the length of OA. Hence,

$$\begin{aligned}
\lim_{\delta t \to 0} r_{\delta t} &= \lim_{\delta t \to 0} \frac{l_{\delta t}}{\sin \delta t} \\
&= \lim_{\delta t \to 0} \frac{a'_S(t_0 + \delta t)}{\sin \delta t} \\
&= \lim_{\delta t \to 0} \left(\frac{a'_S(t_0 + \delta t) - a'_S(t_0)}{\delta t} . \frac{\delta t}{\sin \delta t} \right) \\
&= a''_S(t_0) \ ,
\end{aligned} \tag{6}$$

with this second derivative guaranteed to exist if there is a radius of curvature.

To generalize to the case where $C_S(t_0)$ is other than the origin, let $w(t)$ be the sine wave corresponding to this boundary point. We have $a_S(t_0) = w(t_0) = -w''(t_0)$. Moreover, by shifting S so that $C_S(t_0)$ is moved to the origin, we obtain, by theorem 1, the region with sinusoidal transform $a_S(t) - w(t)$. Therefore the radius of curvature $r_S(t_0)$ we are seeking is given by $a''(t_0) - w''(t_0)$, which immediately yields equation (5).

Kinks in the boundary of S—where the radius of curvature is zero—correspond to intervals over which $C_S(t)$ is constant. But we have established that $a_S(t)$ will be a sine wave over this interval, and so that $a_S(t) + a''_S(t) = 0$, again, giving the correct radius of curvature. Straight line segments in the boundary of S— where the radius of curvature is infinite, correspond to values of t where $a'_S(t)$—and hence $a''_S(t)$—does not exist. But we have established that, at such points, $a_S(t^+) > a_S(t^-)$ so that we might (with a little notational licence) write $r_S(t) = \infty = a_S(t) + a''_S(t)$. The following theorem is a somewhat imprecise summary of the foregoing observations:

Theorem 6 *Let S be a well-behaved, compact, convex region of the plane whose boundary passes through a point P corresponding to a time t (as explained above). Then the radius of curvature of S's boundary at P is given by $a_S(t) + a''_S(t)$.*

We shall derive some further useful properties of the sinusoidal transform below, but this is a good point to pause for some more general analysis. As a preliminary, we note that, if S is well-behaved, and

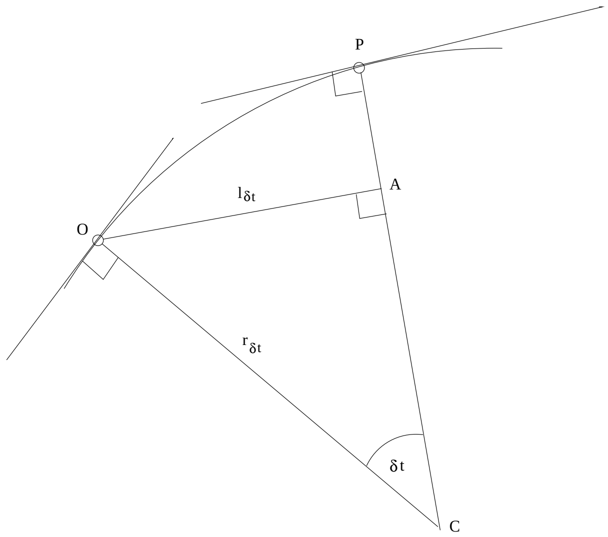


Figure 5: The radius of curvature

$a'_S(t_0)$ exists, then either t_0 is in an interval of pure sine-wave (corresponding to a kink) or $C_S(t_0)$ is neither on a straight-line segment nor at a kink of S's boundary. Either way, $a''_S(t_0)$ exists. Thus, if $a'_S(t)$ exists, so does $a''_S(t)$.

Now consider again equations (3) and (4). It is always possible to solve these equations where $a'_S(t)$ exists, giving

$$x_S(t) = a_S(t)\sin t + a'_S(t)\cos t \quad (7)$$
$$y_S(t) = a_S(t)\cos t - a'_S(t)\sin t\ . \quad (8)$$

But we now know that, for well-behaved regions, if $a'_S(t)$ exists, then so does $a''_S(t)$, which, by equations (7) and (8), guarantees the existence of $x'_S(t)$ and $y'_S(t)$. Routine calculation gives

$$x'_S(t) = (a_S(t) + a''_S(t))\cos t \quad (9)$$
$$y'_S(t) = -(a_S(t) + a''_S(t))\sin t \quad (10)$$

Hence

$$x'_S(t)^2 + y'_S(t)^2 = (a_S(t) + a''_S(t))^2\ . \quad (11)$$

Finally, by the assumption that S is well-behaved, theorem (6) guarantees that, if $C_S(t)$ does not lie on a straight line segment or at a kink in S's boundary,

$$a_S(t) + a''_S(t) > 0\ . \quad (12)$$

This observation will be a useful later. We note also that, for such values of t, $x'_S(t)$ and $y'_S(t)$ cannot both be zero.

We can summarize the properties of sinusoidal transforms of well-behaved regions in the following theorem.

Definition 3 *Let $f(t)$ be a periodic function with period 2π. We say that $f(t)$ is* well-behaved *if $f(t)$ is continuous and, over every period: (i) $f'(t)$ and $f''(t)$ both exist except possibly at a finite number of points t_i such that $f'(t_i^-)$ and $f'(t_i^+)$ both exist with $f'(t_i^+) > f'(t_i^-)$; (ii) whenever $f'(t)$ exists, $f(t) + f''(t) > 0$ except possibly for a finite number of closed intervals $[t'_i, t''_i]$ such that $0 < t''_i - t'_i < \pi$ and $f(t) + f''(t) = 0$ throughout $[t'_i, t''_i]$.*

Theorem 7 *If S is a well-behaved, compact convex region of the plane, then $a_S(t)$ is a well-behaved periodic function with period 2π.*

We shall see in section 4 that a converse of theorem 7 obtains: every well-behaved periodic function with period 2π is the sinusoidal transform of a well-behaved, compact, convex region of the plane.

Now let us return to extracting geometrical information from sinusoidal transforms. The arc-length between any two points on S's boundary can also be simply given in terms of $a_S(t)$. Suppose that S's boundary contains no kinks or straight line-segments at or between the points $C_S(t_1)$ and $C_S(t_2)$. Then the convexity of S will guarantee that that segment of S's boundary will have a well-defined arc-length $\mathrm{arc}_S(t_1, t_2)$. A standard result concerning parametric equations for regular curves tells us that

$$\begin{aligned}\mathrm{Arc}_S(t_1, t_2) &= \int_{t_1}^{t_2} \sqrt{x'^2_S(t) + y'^2_S(t)}dt \\ &= \int_{t_1}^{t_2} (a_S(t) + a''_S(t))dt, \qquad (13)\end{aligned}$$

by equation (11). Note that this result has been established only for intervals corresponding to portions of S's boundary containing no kinks or straight-line segments. We may re-write it as:

$$\mathrm{Arc}_S(t_1, t_2) = \int_{t_1}^{t_2} a_S(t)dt + [a'_S(t)]_{t_1}^{t_2} \ . \qquad (14)$$

If the portion of S's boundary between $C_S(t_1)$ and $C_S(t_2)$ contains kinks strictly between t_1 and t_2, these will correspond to regions of pure sine wave, so that the integral on the right-hand side of equation (13) will be unaffected. Thus, we can drop the requirement that the relevant portion of S's boundary contains no kinks. If the portion of S's boundary between $C_S(t_1)$ and $C_S(t_2)$ contains straight line segments, equation (13) does not make sense, since $a''_S(t)$ does not exist at the times corresponding to those line segments. Surprisingly, however, equation (14), which does make sense, still gives the correct arc length provided that $a''_S(t)$ is defined at t_1 and t_2. To see this, suppose suppose that t_0 corresponds to a line segment in the boundary of S, with $t_1 < t_0 < t_2$. We know that, at t, there will be an instantaneous increase in $a'_S(t)$. Moreover, by equation (2), this line segment is of length $a'_S(t_0^+) - a_S(t_0^-)$. Since the point t_0 makes no contribution to the term $\int_{t_1}^{t_2} a_S(t)dt$ but makes a contribution $a'_S(t_0^+) - a_S(t_0^-)$ to the term $[a'_S(t)]_{t_1}^{t_2}$, the effect on the right-hand side of (14) will be exactly what is needed to take account of the extra arc-length due to the line segment. Thus, we have established:

Theorem 8 *Let S be a well-behaved, compact, convex region, and let t_1, t_2 be times such that $0 \leq t_2 - t_1 \leq 2\pi$ and such that $a''_S(t)$ exists at t_1 and t_2. Then t_1 and t_2 correspond to unique points P_1, P_2, respectively on the boundary of S (in the sense explained above), and the length of S's boundary measured clockwise from P_1 to P_2 is given by:*

$$\mathrm{Arc}_S(t_1, t_2) = \int_{t_1}^{t_2} a_S(t)dt + [a'_S(t)]_{t_1}^{t_2} \ .$$

Theorem 9 *If S is a well-behaved, compact, convex region, its total perimeter is given by:*

$$\mathrm{Perimeter}_S = \int_0^{2\pi} a_S(t)dt \ .$$

The area of S can also be simply expressed in terms of $a_S(t)$. Suppose for the moment that the interior of S contains the origin. And let us assume that the boundary of S contains no straight line segments, so that $a'_S(t)$ and $a''_S(t)$ exist everywhere. Then S's boundary can be described by the periodic function $r(\phi)$, where $(r(\phi), \phi)$ give the polar coordinates of points on that boundary. When using polar coordinates, the standard convention is that ϕ is measured in an anticlockwise direction. In that case, the area of S is given by the standard formula:

$$\text{Area}_S = \frac{1}{2}\int_0^{2\pi} r(\phi)^2 d\phi \ . \tag{15}$$

We can re-parametrize $r(\phi)^2$ in terms of the tangent-angle parameter t as follows:

$$\begin{aligned} r(\phi(t))^2 &= x_S(t)^2 + y_S(t)^2 \\ \phi(t) &= \tan^{-1}\left(\frac{y_S(t)}{x_S(t)}\right) \ . \end{aligned} \tag{16}$$

Note that $x_S(t)$ and $y_S(t)$ cannot both be zero if the origin is interior to S so that $\tan^{-1}$ is well-defined. Moreover, in order for $\phi(t)$ to move from 0 to 2π (measured anticlockwise), t must move backwards from some point t_0 to $t_0 - 2\pi$ (since t is measured in the clockwise direction). Hence

$$\begin{aligned} \text{Area}_S &= \frac{1}{2}\int_0^{2\pi} r(\phi)^2 d\phi \\ &= \frac{1}{2}\int_{t_0}^{t_0-2\pi} r(\phi(t))^2 \frac{d\phi(t)}{dt} dt \\ &= \frac{1}{2}\int_{t_0-2\pi}^{t_0} -\left((x_S(t)^2 + y_S(t)^2)\frac{d}{dt}\left(\tan^{-1}\left(\frac{y_S(t)}{x_S(t)}\right)\right)\right) dt \\ &= \frac{1}{2}\int_0^{2\pi} (-x_S(t)y'_S(t) + y_S(t)x'_S(t))dt \ . \end{aligned}$$

Now, by equations (7) and (8), it is easy to check that

$$-x_S(t)y'_S(t) + y_S(t)x'_S(t) = a_S(t)(a_S(t) + a''_S(t)) \ .$$

Therefore,

$$\text{Area}_S = \frac{1}{2}\int_0^{2\pi} a_S(t)(a_S(t) + a''_S(t))dt \ . \tag{17}$$

Moreover, using the integration-by-parts formula, we can show that, if $w(t)$ is a sine wave of frequency 1,

$$\int_0^{2\pi} w(t)a_S(t)dt = -\int_0^{2\pi} w(t)a''_S(t)dt \ , \tag{18}$$

whence

$$\int_0^{2\pi} (a_S(t) + w(t))(a_S(t) + w(t) + a''_S(t) + w''(t))dt = \int_0^{2\pi} a_S(t)(a_S(t) + a''_S(t))dt \tag{19}$$

so that equation (17) holds even when S does not contain the origin. Finally, we note that, again using integration by parts, we have

$$\int_0^{2\pi} a_S(t)a''_S(t)dt = -\int_0^{2\pi} a'_S(t)^2 dt$$

so that equation (17) becomes

$$\text{Area}_S = \frac{1}{2}\int_0^{2\pi} (a_S(t)^2 - a'_S(t)^2)dt \tag{20}$$

Formula (20) will prove particularly useful in section 5.

Thus we have shown

Theorem 10 *Let S be a well-behaved, compact, convex region of the plane whose boundary contains no line segments. Then the area of S is given by*

$$\text{Area}_S = \frac{1}{2}\int_0^{2\pi} (a_S(t)^2 - a'_S(t)^2)dt \ .$$

The extension of this theorem to cases where S's boundary contains straight line segments is completely routine.

4 Inverting the sinusoidal transform

The job of this section is to answer the following questions:

1. Given a periodic function $f(t)$, of one variable, with period 2π, under what conditions is $f(t)$ the sinusoidal transform of some well-behaved, compact, convex region S_f?
2. For those functions $f(t)$ satisfying these conditions, how can we characterize the mapping from f to S_f (i.e. the inverse of the sinusoidal transform)?
3. Can this inverse mapping be extended to functions which are not the sinusoidal transforms of any convex region? If so, what regions of the plane are obtained?

Let $f(t)$ be any twice differentiable periodic function with period 2π, such that $f(t)+f''(t) > 0$. (We will relax the differentiability requirement and the strict inequality later.) Now define:

$$x_f(t) = f(t)\sin t + f'(t)\cos t \tag{21}$$
$$y_f(t) = f(t)\cos t - f'(t)\sin t \ . \tag{22}$$

By the periodicity of $f(t)$, equations (21) are the parametric equations of a closed curve $C_f = (x_f(t), y_f(t))$ in the plane as t moves from 0 to 2π. We will now show that C_f is simple (does not cross over itself), that the region it encloses is convex and that this region is traversed in a clockwise manner. Differentiating equations (21) and (22) gives, as before

$$x'_f(t) = (f(t) + f''(t))\cos t \tag{23}$$
$$y'_f(t) = -(f(t) + f''(t))\sin t \tag{24}$$

Moreover, since the tangent angle $\theta(t)$ to any parametrically specified curve C_f at $C_f(t)$ is given by

$$\theta(t) = \tan^{-1}\left(\frac{-y'_f(t)}{x'_f(t)}\right) \text{ if } x'_S(t) \text{ and } y'_S(t) \text{ not both zero} \tag{25}$$

straightforward substitution of the equations (23) and (24) yields

$$\theta(t) = t. \tag{26}$$

(We note that $f(t)+f''(t) > 0$ implies that $x'_S(t)$ and $y'_S(t)$ are not both zero, by equations (23) and (24).) Thus, the tangent to C_f at point $C_f(t)$ is t.

By equation (24), $y'_f(t)$ crosses zero at $t = 0$; moreover, since $f(t) + f''(t) > 0$, $y'_f(t)$ is positive just before $t = 0$ and negative just after $t = 0$. Thus, $y_f(t)$ attains a local maximum at $t = 0$, which means that the tangent to C_f at $C_f(0)$ must lie to the left of the curve. By considering shifted versions of the function $f(t)$, we can generalize this to the observation that the tangent to C_f at any point $C_f(t)$ must lie to the left of the curve. In other words, as t moves from 0 to 2π, and we travel around C_f, we are always turning to the right. Since we can only turn through a total of 2π radians, it follows that C_f is simple and encloses a convex region about which it winds clockwise. Note that, if $f(t)$ satisfied the reverse inequality $f(t) + f''(t) < 0$, then parallel reasoning would show that as t moves from 0 to 2π, and we travel around C_f, we would always turn to the left. Thus, in this case too C_f would enclose a convex region, but would travel anti-clockwise around it.

So: given a twice differentiable periodic function $f(t)$ with period 2π such that $f(t) + f''(t) > 0$, equations (21) and (22) generate a compact, convex region R_f. Using these equations together with equations (3) and (4) gives:

$$a_{R_f}(t) \quad = \quad f(t)\ . \tag{27}$$

In other words, the sinusoidal transform of a region R_f generated from $f(t)$ by equations (21) is just $f(t)$. Moreover, given that $f(t) + f''(t) > 0$, we can reverse the derivation (6) in section 3 to conclude that R_f must have a well-defined, positive radius of curvature at every point of its boundary. Hence, R_f is a well-behaved region (one with no kinks or straight line segments in its boundary). Conversely, we know from theorems 6 and 7 that, given a well-behaved compact, convex region S with no kinks or straight line segments in its boundary, its sinusoidal transform $a_S(t)$ is a twice-differentiable periodic function with period 2π such that $a_S(t) + a_S(t)'' > 0$. Moreover, we have $R_{a_S} = S$. In other words, the mappings $S \mapsto a_S$ and $f \mapsto R_f$ are inverses for the case where S is a well-behaved, compact, convex region with no kinks or straight line segments in its boundary, and $f(t)$ satisfies the conditions above.

Having sorted out the correspondence between convex regions and periodic functions where $f''(t)$ always exists and $f(t) + f''(t)$ does not vanish, we now relax these conditions on $f(t)$ slightly. Suppose that $f(t)+f''(t) = 0$ over a subinterval $[t'_i, t''_i]$ of $[0, 2\pi)$ with $0 < t''_i - t'_i < \pi$. By equations (23) and (24), we have $x'_f(t)^2 + y'_f(t)^2 = 0$, so that $x_f(t)$ and $y_f(t)$ are constant over $[t'_i, t''_i]$. This in turn implies the existence of a kink in the curve traced out by C_f, with external angle $t''_i - t'_i$. Thus, C_f traces out a simple closed curve enclosing a convex region R_f as before. Hence, we can weaken the inequality satisfied by $f(t)$ to allow that $f(t)+f(t)'' = 0$ over a finite collection of intervals $[t'_i, t''_i]$ in a period: equation (27) will still hold, of course.

Finally, we consider the case where $f'(t)$ fails to exist at a point t_0, but $f(t)$ is continuous, and the limits $f'(t_0^-)$ and $f'(t_0^+)$ both exist with $f'(t_0^+) > f'(t_0^-)$. Thus, C_f traces out a discontinuous curve with the limits $C_f(t_0^-)$ and $C_f(t_0^+)$ both existing. But we have already agreed that, in such cases, we should take the line from $C_f(t_0^-)$ to $C_f(t_0^+)$ as part of the boundary of S. The continuity of $f(t)$ and the condition $f'(t_0^+) > f'(t_0^-)$ guarantee that the directed line segment from $C_f(t^-)$ to $C_f(t^+)$ will make an angle t with the positive x-axis. Thus, in jumping from $C_f(t^-)$ to $C_f(t^+)$, we continue to turn clockwise as we move around C_f so that C_f is still simple and encloses a convex region R_f. This means that we can weaken the differentiability requirement on $f(t)$ to allow that $f'(t)$ fail to exist at a finite number of points t_i, as long as $f'(t_i^-)$ and $f'(t_i^+)$ both exist and $f'(t_i^+) > f'(t_i^-)$.

Now we can summarize the observations of this section in the following two theorems.

Theorem 11 *If $f(t)$ is a well-behaved, periodic function of one variable with period 2π, then R_f is a well-behaved, compact convex region of the plane.*

Theorem 11 is the promised converse to theorem 7 of section 3. Moreover:

Theorem 12 *For any well-behaved, periodic function $f(t)$ of one variable with period 2π, we have $a_{(R_f)} = f$; and for any well-behaved, compact, convex region S of the plane, we have $R_{(a_S)} = S$.*

Theorems 7, 11 and 12 effectively wrap up the answers to the first two questions set at the beginning of this section. We now know exactly which periodic functions are sinusoidal transforms of well-behaved, convex regions, and, given any such function, equations (21) and (22) enable us to reconstruct the region whose sinusoidal transform it is. But of course, these equations can be used to generate curves from any differentiable periodic function, well-behaved or not. And the question arises as to what happens in the non-well-behaved case. Functions which fail the double-differentiability requirement, or for which $f(t) + f''(t)$ has a minimum at zero are of little computational interest. Moreover, reversing the inequalities and insisting that $f(t) + f''(t) \leq 0$ and that $f(t_i^+) < f(t_i^-)$ generates the same set of regions, but circumnavigated in an anticlockwise direction. The only interesting cases left to understand are (i) when $f(t)+f''(t)$ crosses 0 or (ii) when $f'(t_i^+) \leq f'(t_i^-)$ even though $f(t)+f''(t)$ is not uniformly less than 0.

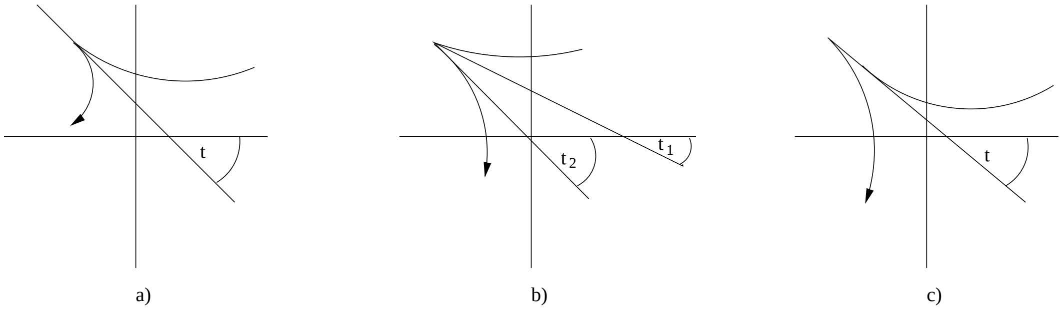


Figure 6: Schematic illustration of various types of cusp in C_f.

Let us assume then, that $f(t)$ is a well-behaved periodic function of one variable, with period 2π, and suppose $f(t) + f''(t)$ crosses 0 at t_0, where $f'(t_0)$ exists. It helps to assume that t does not take one of the values 0, $\pi/2$, π, $3\pi/2$. In that case, from equations (23) and (24), we know that both $x'_f(t)$ and $y'_t(t)$ reverse signs simultaneously, at a point where the tangent to the curve is neither horizontal or vertical. Clearly, this behaviour means that C_f has a cusp at $C_f(t_0)$. This situation is illustrated schematically in figure 6a). Routine checking shows that this result is also true where t_0 is one of 0, $\pi/2$, π, $3\pi/2$. Note that $f(t) + f''(t)$ might linger on the t-axis rather than crossing is instantaneously. In that case, the internal angle of the cusp is equal to the amount of time $f(t) + f''(t)$ spends at 0. This situation is illustrated schematically in figure 6b).

Suppose finally that we have $f'(t_i^+) \leq f'(t_i^-)$ at one of the finitely many points t_i for which $f'(t)$ does not exist. Since $f(t)$ is continuous and differentiable in regions either side of t_i, we have $f'(t_i^+) \neq f'(t_i^-)$. (Otherwise, by a standard result of analysis, $f'(t_i)$ would exist.) Therefore $f'(t_i^+) < f'(t_i^-)$. In other words, we must consider the possibility of an instantaneous *decrease* in the first derivative. As might be expected, under the convention of joining up discontinuities in C_f with straight line segments, this case corresponds to a cusp with a straight line segment inserted in it. Exactly what happens at the cusp depends on the sign of $f(t) + f''(t)$ either side; the details are unsurprising. One such situation—where $f(t) + f''(t)$ changes sign across t_i—is illustrated schematically in figure 6c).

Thus, non-well-behavedness in $f(t)$ leads invariably to cusps in C_f. Since the curve C_f lies to the left or to the right of its tangent according to whether $f(t) + f''(t)$ is positive or negative, there is no reason, in this case, why C_f need trace out a simple curve as t moves though $[0, 2\pi)$. That C_f can fail to be simple when $f(t) + f''(t)$ crosses zero is shown by figure 7 where we have four closed contours, a)–d), corresponding to four arbitrarily chosen, doubly differentiable periodic functions of one variable.

This effectively wraps up our third question concerning the sinusoidal transform, since we now know what happens when we extend the inverse transform to functions which are not the sinusoidal transforms of well-behaved, convex regions. The only geometrically interesting sort of misbehaviour of $f(t)$ arises when $f(t) + f''(t)$ crosses zero or $f(t^+) < f(t^-)$. Such a case represents a cusp, and frequently, though not always, leads to curves C_f which are not simply closed. This (negative) result is of practical interest, because it means that there is no natural extension of the sinusoidal transform which represents any particularly useful class of regions.

5 Fourier series representations

Let us round off with some results of a more positive character. We shall exploit the the sinusoidal transform to represent regions using sinusoidal Fourier descriptors. Let S be a convex region of the

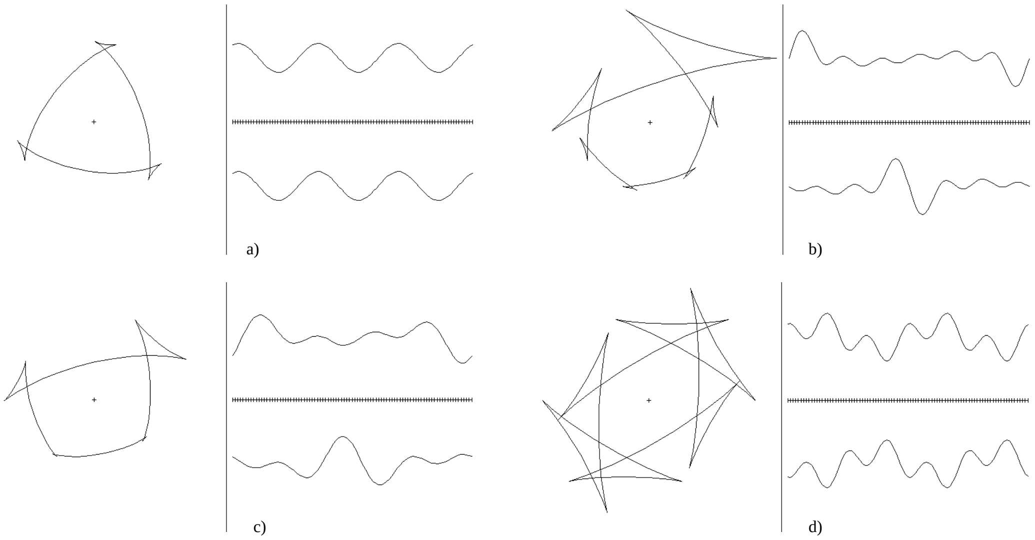


Figure 7: Some functions $f(t)$ for which $f(t) + f''(t)$ crosses zero, and their associated curves $C_f(t) = T(t)F(t)$. Note that the 'lower' function, $-f(t+\pi)$ is also shown.

plane. Since $a_S(t)$ is a periodic function with period 2π, it can be expanded as a Fourier series:

$$a_S(t) = \frac{b_0}{\sqrt{2\pi}} + \sum_{n=1}^{n=\infty} \left(\frac{a_n}{\sqrt{\pi}} \sin nt + \frac{b_n}{\sqrt{\pi}} \cos nt\right)$$

where

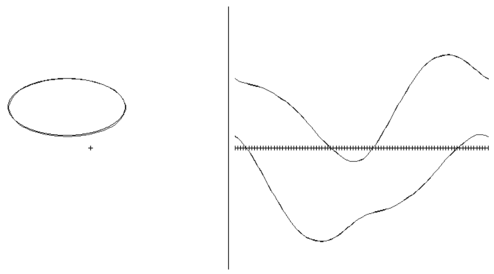
$$\begin{aligned} a_n &= \frac{1}{\sqrt{\pi}} \int_0^{2\pi} a_S(x) \sin nx \; dx \\ b_0 &= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} a_S(x) dx \\ b_n &= \frac{1}{\sqrt{\pi}} \int_0^{2\pi} a_S(x) \cos nx \; dx \; (n > 0) \; . \end{aligned}$$

Thus, instead of thinking of $a_S(t)$ as being the maximum of a collection of sine waves of the same frequency, we now think of it as being a sum of sine waves of different frequencies. Moreover the finite initial segment of the sequence of Fourier coefficients

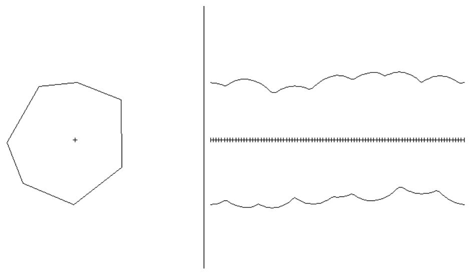
$$b_0, a_1, b_1, a_2, b_2, \ldots a_n, b_n$$

constitutes an approximate (and easily stored) representation of a region S. In practice, a handful of these Fourier coefficients give a convincing representation of most regions. The most noticeable deficits for small values of n are that straight lines tend to be rounded and corners are sometimes replaced by the characteristic fish-tail patters associated with the failure of the condition $f(t) + f''(t) > 0$. Figure 8 shows two regions followed by (an initial segment of) their Fourier descriptors and the reconstructions of these regions from the given Forier descriptors. In the first case, the reconstructed shape is indiscernible from the original. Of course, since $a_S(t)$ and $b_S(t)$ can be recovered from their Fourier descriptors, all of the results derived above can be applied to extract geometric information about the represented regions. Sometimes, however, there are short cuts, as we shall now show.

The first three Fourier descriptors have simple geometric interpretations. From theorem 9, we know that $\sqrt{2\pi} b_0$ is the perimeter length of S. Hence, b_0 gives a measure of the size of S. Moreover, the



Original



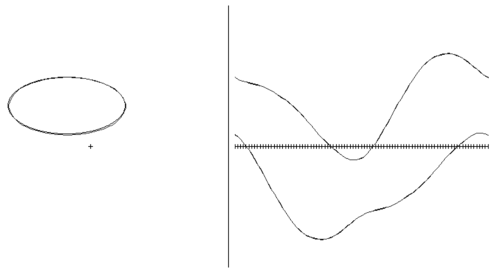
Original

n	a_n	b_n
0		47.8315
1	-35.0	61.0
2	0.0	-22.0
3	0.0	0.0
4	0.0	-1.8429
5	0.0	0.0
6	0.0	-0.7579

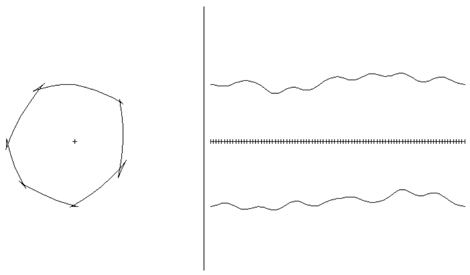
Fourier descriptors

n	a_n	b_n
0		63.4020
1	-9.3731	-1.2317
2	3.1498	1.0784
3	0.7254	-2.0080
4	-2.1575	-0.7121
5	-2.0217	-1.9128
6	-1.8057	0.7393
7	2.4213	-0.9125
8	0.3109	0.4162
9	-0.4445	0.4782
10	0.3803	0.2867

Fourier descriptors



Reconstruction



Reconstruction

a) b)

Figure 8: Two regions and their reconstructions using a finite number of Fourier descriptors.

Fourier coefficients $(a_1/\sqrt{\pi}, b_1/\sqrt{\pi})$ define a sine wave of frequency 1—in other words a point in the plane. From theorem 1, it is easy to see that this point is rigidly connected to S in the sense that, if S is rotated and translated, the point defined by $(a_1/\sqrt{\pi}, b_1/\sqrt{\pi})$ will be rotated and translated inexactly the same way. Hence, $(a_1/\sqrt{\pi}, b_1/\sqrt{\pi})$ indicates the position of S. We remark that $(a_1/\sqrt{\pi}, b_1/\sqrt{\pi})$ is not in general the centroid of S. Finally, since all circles have $a_k = b_k = 0$ for all $k > 1$, the higher-order Fourier coefficients alone are responsible for any eccentricity. However, it is hard to be more specific about the meaning of these higher-order coefficients.

Formula (20) for computing areas of convex regions also has a simple formulation in terms of the Fourier descriptors. By Parseval's formula,

$$\begin{aligned}\int_0^{2\pi} a_S(t)^2 dt &= b_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \\ \int_0^{2\pi} a'_S(t)^2 dt &= \sum_{n=1}^{\infty} n^2(a_n^2 + b_n^2)\ .\end{aligned}$$

Therefore,

$$\text{Area}_S = \frac{1}{2}\left(b_0^2 - \sum_{n=2}^{\infty}(n^2-1)(a_n^2 + b_n^2)\right)\ .$$

The geometrical interpretation of b_0 and (a_1, b_1) simplifies the problem of determining congruence and similarity. The following theorem is easily verified:

Theorem 13 *Let S and S' be well-behaved, compact, convex regions whose sinusoidal transforms are defined by the Fourier descriptors*

$$\begin{aligned}&b_0, a_1, b_1, a_2, b_2, \ldots \\ &b'_0, a'_1, b'_1, a'_2, b'_2, \ldots\end{aligned}$$

respectively. Then S and S' are congruent if and only if

$$\begin{aligned}b_0 &= b'_0 \\ {a_n}^2 + {b_n}^2 &= {a'_n}^2 + {b'_n}^2\end{aligned}$$

and

$$\frac{1}{n}\left(\tan^{-1}\left(\frac{b_n}{a_n}\right) - \tan^{-1}\left(\frac{b'_n}{a'_n}\right)\right)$$

is constant for $n \geq 2$. Moreover, S and S' are similar if, in addition, the above result holds after normalizing so that $b_0 = b'_0$.

This theorem of course only yields conditions for *exact* congruence and similarity, whereas, in practice, we need to know whether two regions are *nearly* congruent or similar. In fact, sinusoidal Fourier coefficients are relatively stable in the sense that small variations in the figure lead to only small variations in the sinusoidal Fourier coefficients that represent it. However, we shall not pause to investigate these issues here, or to develop error-tolerant tests for similarity and congruence.

6 Summary of Formulae

- Rotation and translation:

$$\begin{aligned}a_{S_{\vec{v}}}(t) &= a_S(t) + w_{\vec{v}}(t) \\ a_{S_\theta}(t) &= a_S(t+\theta)\ .\end{aligned}$$

- Overlap and inclusion:

$$S \cap T \neq \emptyset \text{ iff } \forall t\ a_T(t) \geq b_S(t)$$
$$S \subseteq T \text{ iff } \forall t\ a_T(t) \geq a_S(t).$$

- Separation and distance:

$$\begin{aligned} d(S,T) &= \sup(\{(b_T(t) - a_S(t)) | t \in [0, 2\pi)\}) \\ \Delta(S,T) &= \sup(\{0\} \cup \{(a_S(t) - a_T(t)) | t \in [0, 2\pi)\}) + \sup(\{0\} \cup \{(a_T(t) - a_S(t)) | t \in [0, 2\pi)\}) \ . \end{aligned}$$

- Radius of curvature:

$$r_S(t) = a_S(t) + a''_S(t) \ .$$

- Arc-length:

$$\mathrm{Arc}_S(t_1, t_2) = \int_{t_1}^{t_2} a_S(t)dt + [a'_S(t)]_{t_1}^{t_2} \ .$$

- Perimeter:

$$\begin{aligned} \mathrm{Perimeter}_S &= \int_0^{2\pi} a_S(t)dt \\ &= \sqrt{2\pi} b_0 \ . \end{aligned}$$

- Area:

$$\begin{aligned} \mathrm{Area}_S &= \frac{1}{2}\int_0^{2\pi} (a_S(t)^2 - a'_S(t)^2)dt \\ &= \frac{1}{2}(b_0^2 - \sum_{n=2}^{\infty}(n^2-1)(a_n^2 + b_n^2)) \ . \end{aligned}$$

- Congruence:

  S and S' are congruent if and only if

$$\begin{aligned} b_0 &= b'_0 \\ {a_n}^2 + {b_n}^2 &= {a'_n}^2 + {b'_n}^2 \end{aligned}$$

  and

$$\frac{1}{n}\left(\tan^{-1}\left(\frac{b_n}{a_n}\right) - \tan^{-1}\left(\frac{b'_n}{a'_n}\right)\right)$$

  is constant for $n \geq 2$.

7 Conclusions

The primary achievement of this paper has been to gain a better understanding of a simple and intuitively appealing method for representing convex regions of the plane. We demonstrated that the sinusoidal transform of any (reasonable) convex region of the plane is a periodic function $f(t)$ for which $f(t) + f''(t)$ has no zero-crossings. We demonstrated, conversely, that any (reasonable) periodic function $f(t)$ for which $f(t) + f''(t)$ has no zero-crossings is the sinusoidal transform of a convex region. Furthermore, we showed that the natural extension of this correspondence to cases where $f(t) + f''(t)$ changes sign yields only cusped regions or curves which are not even simply closed. In particular, there is no room in the space of periodic functions of one variable for representing concave regions with points of inflexion. Finally, as a result of our analysis, we showed that various geometrical quantities and relations—overlap, inclusion, separation, distance, radius of curvature, perimeter length, area and congruence—could be extracted from the sinusoidal transforms of regions, and hence from their Fourier descriptors.

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