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Abstract

Several authors have suggested that a more parsimonious and conceptually elegant treatment of everyday mereological and topological reasoning can be obtained by adopting a spatial ontology in which regions, not points, are the primitive entities. This paper challenges this suggestion for mereotopological reasoning in 2-dimensional space. Our strategy is to define a mereotopological language together with a familiar, point-based interpretation. It is proposed that, to be practically useful, any alternative region-based spatial ontology must support the same sentences in our language as this familiar interpretation. This proposal has the merit of transforming a vague, open-ended question about ontologies for "practical" mereotopological reasoning into a precise question in model theory. We show that (a version of) the familiar interpretation is countable and atomic, and therefore prime. We conclude that useful alternative ontologies of the plane are, if anything, less parsimonious than the one which they are supposed to replace.

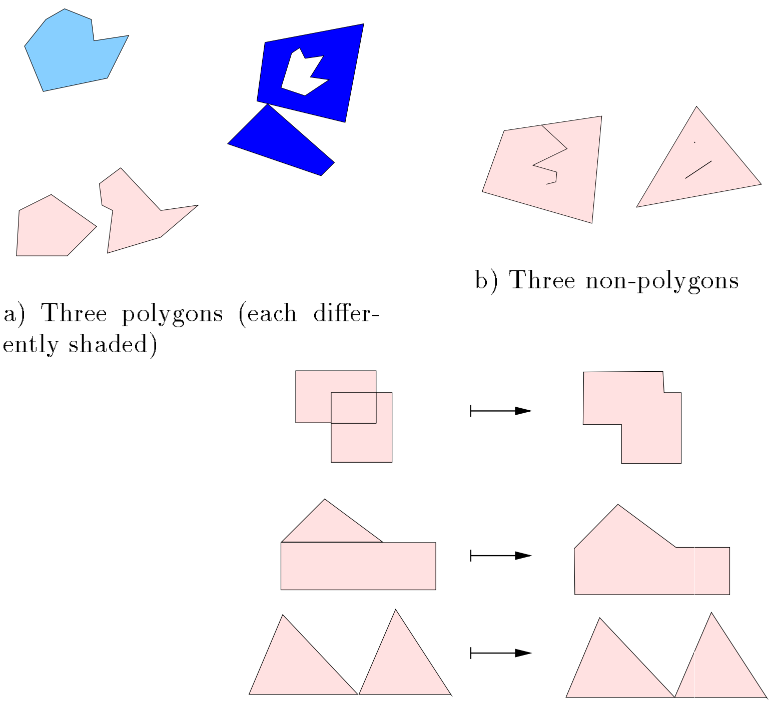
1 The problem

One of the many achievements of coordinate geometry has been to provide a conceptually elegant and unifying account of the nature of geometrical entities. According to this account, the one primitive spatial entity is the point, and the one primitive geometrical property of points is coordinate position. All other geometrical entities—lines, curves, surfaces and bodies—are nothing but collections of points; and all properties and relations involving these entities may be defined in terms of the relative positions of the points which make them up. The success and power of this reduction is so great that the identification of spatial regions with the sets of points which they contain has come to seem virtually axiomatic.

Yet various authors have sought to reverse this order of rational reconstruction, treating *regions* as primary, and admitting points, if at all, as logical constructions out of them. The best known of these approaches is perhaps Tarski's [36] axiomatization of Euclidean geometry, taking spheres to be the primitive entities. But the policy of taking regions as primitive is most attractive when considering problems involving mereological (part-whole) and topological notions—that is, where no metric information is to hand. If regions are first class entities and points are logical constructions based on them, then who knows what interesting new ways of considering spatial entities and relations there might be? Clarke [12], [13], following an idea of Whitehead [40], sought to reconstruct mereotopology in terms of a primitive relation of connection holding between regions. Following this work, Biacino and Gerla [5] have studied models of Clarke's theory. More recently, and partially as a response to debates concerning temporal reasoning and knowledge representation (for example, Allen [1]), Clarke's mereotopology has received attention from several research groups in AI, working with the loosely defined area of *qualitative spatial reasoning*, for example, Gotts, Gooday and Cohn [18], Asher and Vieu [2], and Borgo, Guarino, and Masolo [7].

Motivations for these developments vary, and we do not intend to provide a comprehensive account of them here. However, one common recurring theme is the suspicion that the familiar, point-based, view of space generates a richer ontology than is needed for mereotopological reasoning in "practical" situations. For example, Euclidean space contains not only the sorts of regions we want to recognize for everyday purposes, but also strange, physically unrealizable regions of the kind that populate point-set topology textbooks. Such regions seem to be mere artifacts of the Euclidean model of space—useless for describing, and reasoning about, the world we inhabit. If, on the other hand, we regard regions as primitive entities, perhaps we can be more selective as to what regions we take to exist and what mereotopological properties we take them to have. Perhaps—so some researchers in mereotopology suggest—treating regions as primary opens up the prospect of simpler and more parsimonious spatial ontologies than the familiar model based on points in the real plane.

The present paper examines this suggestion for the special case of *plane* mereotopology. We show that, under certain reasonable assumptions as to what practical mereotopological reasoning might involve, taking regions rather than points as primitive cannot lead to a more parsimonious spatial ontology.



a) Three polygons (each differently shaded)

b) Three non-polygons

c) Three pairs of polygons and their respective sums

Figure 1: Polygonal mereotopology

2 Polygonal mereotopology

To get an idea of what "practical" mereotopological reasoning might involve, consider computer systems specialized for representing plane spatial data, such as, for example, Geographic Information Systems (GISs). Virtually all such systems represent regions of space by means of boundaries consisting of finitely many straight lines and straight-line segments. In effect, then, all plane regions recognized by such systems are polygons. Experience has shown that such a spatial ontology, whatever its philosophical shortcomings, is certainly equal to the task of describing everyday planar spatial arrangements such as those found on maps and charts, since every arrangement of regions one is likely to encounter can be approximated by polygons with arbitrarily high accuracy.

Suppose, then, we take as our spatial ontology the set P of polygons in the plane. We discuss the formal construction of P later; for the present, all that matters is that all members of P are the interiors of plane regions bounded by finitely many straight lines, as shown in figure 1a. Note that we allow polygons to consist of more than one piece, and to contain holes (as long as those holes have straight-line boundaries); however, polygons are not allowed to contain "cracks", as shown in figure 1b. In addition, we consider the empty set and the whole plane to be polygons.

It turns out that P forms a Boolean algebra. In this Boolean algebra, the product of two polygons is their intersection; the negation of a polygon is that part of the plane lying outside it and its boundary; and the sum of two polygons is the polygon formed by taking their union and 'rubbing out' any internal boundaries that result. Figure 1c illustrates the sum-operation. Accordingly, our mereotopological language will be equipped with functions-symbols $\cdot$, $-$ and $+$ to denote these operations, as well as the constants 0 and 1 to denote the empty set and the whole plane, respectively. Note that the formula $x.y = x$ states that x is a subset of y. Hence the Boolean functions can express various mereological properties and relations involving polygons.

In point-set topology, it is usual to define an open set as being *connected* if it is not the union of two disjoint, nonempty, open sets. Intuitively, connected sets are just that—they consist of one piece. Accordingly, our language will be equipped with a one-place predicate $c(x)$ to express the property of being a connected polygon. (Incidentally, since we are concerned only with *open* sets, the right-most polygon in figure 1a is not connected.) If x and y are disjoint, connected and non-empty, then it is possible to show that the formula $c(x+y)$ is satisfied if and only if x and y share one or more proper straight line segments on their boundaries. In other words, the formula $c(x+y)$ can be used to express the relation of external contact along an edge. Hence, the predicate $c(x)$, together with the Boolean functions, can express various topological properties and relations involving polygons.

Thus, we take our mereotopological language $\mathcal{L}$ to be a first-order language with equality and non-logical constants $+$, $\cdot$, $-$, 0, 1 and $c(x)$. The set P of polygons will form the domain over which the variables of $\mathcal{L}$ range, and the interpretation of the non-logical constants of $\mathcal{L}$ given above defines a model $\mathfrak{P}$ on the domain P. The sentences $\mathrm{Th}(\mathfrak{P})$ true in this model represent, as it were, the facts of mereotopology according to the the polygonal ontology employed in most computer systems for representing plane spatial data.

We propose to take $\mathrm{Th}(\mathfrak{P})$ to be the facts of "practical" mereotopological reasoning. After all, the polygonal model $\mathfrak{P}$ is relatively simple, admits no pathological regions, and yet is mereotopologically non-trivial and finds use in many practical applications without apparent loss of useful representational power. We further propose that an alternative spatial ontology for practical mereotopological reasoning is simply an alternative model of $\mathrm{Th}(\mathfrak{P})$—that is, a model $\mathfrak{A}$ such that $\mathfrak{A} \equiv \mathfrak{P}$ but $\mathfrak{A} \not\simeq \mathfrak{P}$. The domain A of $\mathfrak{A}$ will form the set of regions of space and the relations and properties needed to interpret the terms in $\mathcal{L}$ will give this space its mereotopological structure.

Note that the domain P contains only polygonal *regions*, and not the points and lines of which they are made up. Thus, we employ a language which can talk, in the first instance, only about regions, in keeping with the spirit of mereotopology. On the other hand, our model $\mathfrak{P}$ is fundamentally Euclidean, in that polygons are objects in the Euclidean plane defined in terms of the points they contain or the lines that bound them. Thus, $\mathfrak{P}$ is, as it were, our *familiar* ontology—one constructed in the familiar way from points in the Euclidean plane. A *general* model of $\mathrm{Th}(\mathfrak{P})$, by contrast, may have any sorts of objects in its domain, either primitive or constructed in some other way. The problem we face in the sequel is to identify such general models, and to determine whether any of them constitute a more elegant and parsimonious spatial ontology than $\mathfrak{P}$.

It may be objected that this strategy is too conservative. After all, *who says* that the facts of practical mereotopological reasoning—the facts that we would want any alternative spatial ontology to support—are the facts that are true of the polygonal data-structures employed in many computer systems? Perhaps we could find a better theory of space by *revising* this "computer-mereotopology". To some extent, this criticism is justified: we have little to say in favour of the polygonal theory, except that it is familiar, easily formalized and seems to be widely and successfully used in a vast range of practical applications. (Proponents of other theories of mereotopological reasoning should be so lucky.) Nevertheless, our strategy does have the virtue of transforming a vague and open-ended question about ontologies for "practical" mereotopological reasoning into a precise, technical question in model theory. To be sure, we do not regard the solution of this problem to be the last word on the metaphysics of space; but at least we now have a definite question to address.

3 Regular sets

Before any serious analysis begins, we must formalize the familiar model $\mathfrak{P}$ and our mereotopological language $\mathcal{L}$. We do this in two stages: the present section establishes some basic groundwork at a rather high level of generality; the next section goes on to present the details of $\mathfrak{P}$ and $\mathcal{L}$.

Our first task is to resolve the issue of whether regions include their boundary points. We adopt an approach, based on *regular sets*, which has become reasonably standard in discussions of spatial description languages.

Definition 3.1 *Let X be a topological space and $x \subseteq X$. Then the set $\bigcup\{y \subseteq X \mid y \text{ open}, y \cap x = \emptyset\}$ is an open set in X called the* pseudocomplement of X, *written x'. We say that $x \subseteq X$ is* regular *if*

$x = x''$.

Obviously, $x \cap x' = \emptyset$. Hence, if x is open, then $x \subseteq x''$.

The following well-known theorem underlies the importance of the regular sets to mereotopology. We state it here without proof. (See, e.g. Johnstone [21], section 1.13.)

Theorem 1 *Let X be a topological space. Then the set of regular sets in X forms a Boolean algebra with top and bottom defined by $1 = X$ and $0 = \emptyset$, and Boolean operations defined by $x.y = x \cap y$, $x + y = (x \cup y)''$ and $-x = x'$.*

Accordingly, we shall sometimes use the term *regular Boolean algebra* of a topological space X to refer to the Boolean algebra of regular sets of X. We denote this algebra by $M(X)$. When dealing with the elements of such a Boolean algebra, we shall write $x.y$, $x + y$, $-x$ and instead of $x \cap y$, $(x \cup y)''$ and x', respectively.

There is an alternative characterization of the regular sets. If X is a topological space and $y \subseteq X$ is any set, then there will always be a largest open set contained in y, called the *interior* of y, and denoted by y^0. Likewise, there will always be a smallest closed set containing y, called *the closure* of y, and denoted by $[y]$. The set $[x] \setminus ([x])^0$ is called the *frontier* of x, and is denoted by $\mathcal{F}(x)$.

Lemma 3.1 *Let X be a topological space and $x \subseteq X$. Then $x' = X \setminus [x]$.*

Proof: $X \setminus [x] = X \setminus \bigcap\{u \mid u \subseteq X$ closed and $x \subseteq u\} = \bigcup\{X \setminus u \mid u \subseteq X$ closed and $x \subseteq u\} = \bigcup\{y \subseteq X \mid y$ open and $x \cap y = \emptyset\} = x'$. □

Lemma 3.2 *Let x be a subset of a topological space X. Then $x'' = ([x])^0$.*

Proof: Certainly, $([x])^0$ is open and is a subset of $[x] = X \setminus x'$. Then $x' \cap ([x])^0 = \emptyset$ so $([x])^0 \subseteq x''$. Conversely, $x' \cap x'' = \emptyset$ so $x'' \subseteq X \setminus x' = [x]$. Since x'' is open and $x'' \subseteq [x]$, $x'' \subseteq ([x])^0$. □

Hence, a set in a topological space is regular if and only if it is equal to the interior of its closure.

Theorem 1 and lemma 3.2 show that the regular sets of a topological space form a natural domain of quantification for a mereotopological language. First, theorem 1 shows that the part-whole relationship, restricted to the regular sets, still obeys the axioms of a Boolean algebra, and so will be mathematically manageable. Second, lemma 3.2 guarantees that boundary points are ignored, in the sense that no two regular regions differ only with respect to their boundary points, and that all non-regular regions differ from regular regions only with respect to their boundary points.

We conclude this section with two lemmas concerning regular sets: both will be used in the results derived below.

As usual in topology, we say that an open set x is *connected* if there do not exist two nonempty, disjoint open sets whose union is x.

Lemma 3.3 *Let a_1, a_2, a_3 be regular sets of a topological space X with $a_1 + a_2, a_2 + a_3$ connected and $a_2 \neq 0$. Then $a_1 + a_2 + a_3$ is connected.*

Proof: By a standard result, if x is any connected set and $x \subseteq y \subseteq [x]$ then y is connected. Since $a_2 \neq 0$, $(a_1 + a_2) \cup (a_2 + a_3)$ is certainly connected. And $(a_1 + a_2) \cup (a_2 + a_3) \subseteq (a_1 + a_2) + (a_2 + a_3) = ((a_1 + a_2) \cup (a_2 + a_3))'' \subseteq [(a_1 + a_2) \cup (a_2 + a_3)]$, by lemma 3.2 □

Lemma 3.4 *Let X, Y be topological spaces and ν a homeomorphism from X onto Y. Let a, b be regular sets in X. Then $\nu(a)$ and $\nu(b)$ are regular sets in Y with: (i) $\nu(a.b) = \nu(a).\nu(b)$; (ii) $\nu(-a) = -\nu(a)$; and (iii) $\nu(a + b) = \nu(a) + \nu(b)$.*

For ease of notation, if x and y are open, we shall write $x.y$ for $x \cap y$ and $-x$ for x', even when x and y are not known to be regular.

Proof: Equation (i) holds because ν is 1–1. Equation (ii) follows if we can show that $\nu(-a) \subseteq -\nu(a)$ and $-\nu(a) \subseteq \nu(-a)$. By equation (i), $\nu(a).\nu(-a) = \nu(\emptyset) = \emptyset$. Since ν^{-1} is continuous, $\nu(a)$ and $\nu(-a)$ are open. Hence $\nu(-a) \subseteq -\nu(a)$. To prove the reverse inclusion, it suffices to show that, for any open set $z \subseteq Y$ satisfying $z.\nu(a) = \emptyset$, $z \subseteq \nu(-a)$. Let z be such a set. Since $z.\nu(a) = \emptyset$, it follows that $\nu^{-1}(z).a = \emptyset$. And since ν is continuous, $\nu^{-1}(z)$ is open. Therefore $\nu^{-1}(z) \subseteq -a$, so $z \subseteq \nu(-a)$. Equation (iii) follows from (i) and (ii).

Finally, if a is regular, by equation (ii), $- - \nu(a) = \nu(- - a) = \nu(a)$, so $\nu(a)$ is regular. □

4 The polygonal models

Having dealt with the very general notions of regular sets and their Boolean algebras, we return in this section to the polygonal domain.

Any line in the Euclidean plane, $\mathbb{R}^2$, cuts $\mathbb{R}^2$ into two connected, open sets, called *half-planes*. It is easy to see that these sets are regular, with each being the pseudocomplement of the other. Hence, we can speak about the sums, products and complements of half-planes in $M(\mathbb{R}^2)$.

Definition 4.1 *A* basic polygon *is the intersection of finitely many half-planes in* $\mathbb{R}^2$.
A polygon *is the sum, in* $M(\mathbb{R}^2)$*, of any finite set of basic polygons.*

We denote the set of polygons by R, and will sometimes refer to it as the *polygonal domain*. Thus, the elements of R are simply polygons as introduced in the previous secion.

Of course, R is not the only well-behaved spatial domain we might choose. If a line is defined by an equation $ax + by + c = 0$, where a, b and c are rational numbers, we call it a *rational line*; and if a half-plane is bounded by a rational line, we call it a *rational half-plane*. It is obvious that a pair of rational lines can intersect only at points with rational coordinates. Now we define:

Definition 4.2 *A* rational basic polygon *is the intersection of finitely many rational half-planes.*
A rational polygon *is the sum, in the Boolean algebra of regular sets of the plane, of any finite set of rational basic polygons.*

We denote the set of rational polygons by Q, and will sometimes refer to it as the *rational polygonal domain*. Thus, R, or perhaps, more modestly, Q, is the spatial ontology recognized by computer systems such as GISs—both domains provide a simple view of space from which any remotely pathological behaviour has been excluded. Clearly, R is uncountable, whereas Q is countable; so R and Q are different structures. Nevertheless, their ontologies are very similar, and share many basic properties. For brevity, we use the symbol P to denote either R or Q.

Now that we have defined the polygonal domain (or, more precisely, domains), P, we introduce our mereotopological language $\mathcal{L}$. Let $\mathcal{L}$ be the first-order language with equality having the following non-logical symbols:

1. one 1-place predicate $c(x)$
2. two constant symbols 0 and 1
3. one 1-place function symbol $-x$
4. two 2-place function symbols $x + y$ and $x.y$

Informally, the functions $+$, . and $-$ correspond to operations in the Boolean algebra $M(\mathbb{R}^2)$; and $c(x)$ means that x is connected (in the usual topological sense). Thus, $\mathcal{L}$ has a mereological component in the form of Boolean connectives representing operations on regular sets, and a topological component in the form of a connectedness predicate.

Formally, we give $\mathcal{L}$ two 'familiar' interpretations, $\mathfrak{R}$ and $\mathfrak{Q}$, corresponding to the domains R and Q, respectively. First, we must check that the Boolean operations of $M(\mathrm{I\!R}^2)$ can be applied freely in these domains.

Theorem 2 *P is a Boolean subalgebra of $M(\mathrm{I\!R}^2)$.*

Proof: We need only show that P is closed under the Boolean operations. But this is obvious given the distribution laws for $M(\mathrm{I\!R}^2)$. □

Definition 4.3 *We define the* polygonal model $\mathfrak{R}$ *to have the domain R and the following interpretations of the predicate, constant and function symbols in $\mathcal{L}$:*

1. $c^{\mathfrak{R}}(x) = x$ is connected
2. $0^{\mathfrak{R}} = \emptyset$; $1^{\mathfrak{R}} = \mathrm{I\!R}^2$
3. For all $a \in R$: $-^{\mathfrak{R}}(a) = -a$
4. For all $a, b \in R$: $+^{\mathfrak{R}}(a,b) = a + b$ and $.^{\mathfrak{R}}(a,b) = a.b$

We define the rational polygonal model $\mathfrak{Q}$ *exactly as for $\mathfrak{R}$ but with $\mathfrak{R}$ and R replaced throughout by $\mathfrak{Q}$ and Q respectively.*

Again, in view of the similarities between $\mathfrak{R}$ and $\mathfrak{Q}$, we write $\mathfrak{P}$ to refer indeterminately to either. Thus, the domain of $\mathfrak{P}$ is P. Anticipating a result of the next section, it turns out—unsurprisingly—that $\mathfrak{R}$ and $\mathfrak{Q}$ make the same sentences of $\mathcal{L}$ true. That is, the ontologies $\mathfrak{Q}$ and $\mathfrak{R}$ are indistinguishable for the mereotopological language $\mathcal{L}$. Hence we may write $\mathrm{Th}(\mathfrak{P})$ to denote $\mathrm{Th}(\mathfrak{R}) = \mathrm{Th}(\mathfrak{Q})$. Our main task in this paper is to find alternative models of $\mathrm{Th}(\mathfrak{P})$.

We finish this section on the familiar models for $\mathcal{L}$ with an example to show that the pains we took to define our domain of interpretation were not in vain. Consider the following formula of $\mathcal{L}$:

$$\forall x_1 \forall x_2 \forall x_3 \left(\left(\bigwedge_{1 \leq i \leq 3} c(x_i) \wedge c(x_1 + x_2 + x_3) \right) \rightarrow (c(x_1 + x_2) \vee c(x_1 + c_3)) \right). \tag{1}$$

This formula asserts that, if the sum of three connected regions is connected, then the first must be connected to at least one of the other two. It is true in the model $\mathfrak{P}$; but it would be false in a model whose domain extended to all regular sets of the plane. For consider the regions a_1, a_2 and a_3 defined by

$$\begin{aligned} a_1 &= \{(x,y) | -1 < x < 0 \ ; \ -1 - x < y < 1 + x\} \\ a_2 &= \{(x,y) | 0 < x < 1 \ ; \ -1 - x < y < \sin(1/x)\} \\ a_3 &= \{(x,y) | 0 < x < 1 \ ; \ \sin(1/x) < y < 1 + x\} \ , \end{aligned}$$

and depicted in figure 2. (Note: in this figure, the x-axis has been dilated.) It is not difficult to show that a_1, a_2 and a_3 are regular, that $a_1 + a_2 + a_3$ is the interior of the large triangle in figure 2 and so is connected, but that neither $a_1 + a_2$ nor $a_1 + a_3$ is connected.

Thus, for the purposes of practical mereotopological reasoning, (1) is quite a sensible formula to have as a theorem of our calculus, since—as it turns out—the only counterexamples are rather pathological and seem to be artifacts of the Euclidean model of space rather than anything that one could come across in real life. (Adopting such a stance is very much within the spirit of mereotopological research in AI,

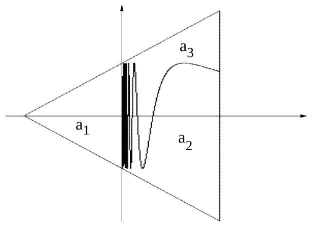


Figure 2: Three regular sets in the plane

where the ability to reason efficiently in everyday situations is in focus.) But this case does demonstrate the importance of having a precise characterization of the regions our mereotopological language talks about. When looking for models elementarily equivalent to $\mathfrak{P}$ as alternative ontologies for practical mereotopological reasoning, we are making some very specific choices about the facts of mereotopology that we want to support.

5 Topological analysis

Our next step is to establish some basic topological properties of P. The most important of these properties are described by theorem 4 and lemmas 5.9 and 5.10. All other results in this section are ancillary.

We begin with a lemma on which much of the subsequent analysis depends.

Lemma 5.1 *Any element of P is the sum of finitely many connected elements of P.*

Proof: Since half-planes are convex, basic polygons are convex, and so are certainly connected. □

It easy to see that this property does not hold for all Boolean sub-algebras of $M(\mathbb{R}^2)$, even where the elements are relatively well-behaved. For example, even when x and y are Jordan regions (i.e. topologically equivalent to the unit circle), the intersection $x.y$ may have infinitely many disconnected parts. It is precisely to prevent this possibility that we shall restrict attention to the domains R and Q. To be sure, the polygons are not the only set of regions satisfying lemma 5.1, but they are the simplest.

As usual, we take a *component* of a set to be a maximal, nonempty connected subset of that set.

Lemma 5.2 *Let $a \in P$ and let c be a component of a. Then $c \in P$. Moreover, a equals the sum of its components.*

Proof: By lemma 5.1, let $c_1, \ldots, c_n$ be connected elements of P such that $a = c_1 + \ldots + c_n$. For all i $(1 \leq i \leq n)$, if $c.c_i \neq 0$ then, by lemma 3.3, $c_i + c$ is connected. If, in addition, $(-c).c_i \neq 0$, then $c < c + c_i$, contradicting the maximality of c. Thus, if $c.c_i \neq 0$, then $(-c).c_i = 0$. Hence c can be expressed as the sum of various c_i $(1 \leq i \leq n)$, and $c \in P$. The remainder of the lemma is trivial. □

Let M be any Boolean subalgebra of $M(X)$. If A is a finite subset of M and the elements of A are pairwise disjoint, nonempty and sum to $a \in M$, we call A a *partition of* a in M. If, in addition, every element of A is connected, we call A a *connected partition of* a in M. In the case $a = 1$, we refer to A, simply, as a (*connected*) *partition* in M. The following (rather technical) lemma will be useful later.

Lemma 5.3 *Let X be a topological space, M a Boolean subalgebra of $M(X)$ and $a_1, \ldots, a_n$ a partition in M. Let m be such that $1 \leq m \leq n$. Then*

$$a_1 + \ldots + a_m = a_1 \cup \ldots a_m \cup \{p |\ p \in \mathcal{F}(a_i) \text{ for some } i \ (1 \leq i \leq m),$$
$$p \notin \mathcal{F}(a_j) \text{ for any } j \ (m < j \leq N)\}.$$

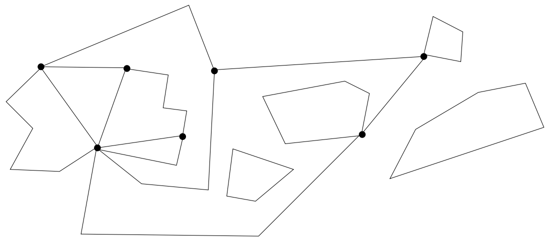


Figure 3: A graph* with two nodeless edges

Proof: Denote the right hand side of the above equation by x. Suppose $p \in [a_j]$ for some j $(m < j \leq n)$. Then $p \in \mathcal{F}(a_j)$ or $p \in ([a_j])^0 = a_j$ by lemma 3.2. If $p \in a_j$, then $p \notin [a_i]$ for any i $(1 \leq i \leq m)$ by the disjointness of $a_1, \dots, a_n$. Either way, then, $p \notin x$.
Suppose $p \notin [a_j]$ for any j $(m < j \leq n)$. Then certainly $p \notin \mathcal{F}(a_j)$ for any j $(m < j \leq n)$. Moreover, $a_1, \dots, a_n$ sum to 1, so $[a_1] \cup \dots \cup [a_n] = 1$. Hence $p \in [a_i]$ for some i $(1 \leq i \leq m)$, so, again, $p \in \mathcal{F}(a_i)$ or $p \in ([a_i])^0 = a_i$ by lemma 3.2. Either way, $p \in x$.
Hence $x = (1 \setminus [a_{m+1}]). \dots .(1 \setminus [a_n])$. By lemma 3.1, $x = (-a_{m+1}). \dots .(-a_n) = a_1 + \dots + a_m$. □

Connected partitions play an important role in understanding R and Q. In particular, we have:

Lemma 5.4 *Let A be a finite subset of P. Then there exists a connected partition C of P such that each $a \in A$ is expressible as a sum of various $c \in C$.*

Proof: If $A = \{a_1, \dots, a_n\}$, let C be the set of all components of all non-zero products of the form $\pm a_1. \dots . \pm a_n$. By lemma 5.2, these components are elements of P, and form a connected partition such that every a_i can be expressed as a sum of various C. □

Furthermore, it should come as no surprise that we can picture connected partitions in P by thinking in terms of plane graphs.

Definition 5.1 *A* graph* *G is a plane graph in the closed real plane having no nodes of degree 0, together with a (possibly empty) set of* nodeless edges*. These nodeless edges are all Jordan curves intersecting no other edge of G (nodeless or otherwise).*
A graph is* piecewise linear *if all of its edges lie on finitely many straight lines; a graph* is* rational piecewise linear *if all of its edges lie on finitely many rational straight lines.*

Hence, all plane graphs in the closed plane are graphs*. Figure 3 shows a piecewise linear graph* (where the page represents the whole plane) with two nodeless edges. This particular specimen also has no isthmuses and no nodes of degree 2. We note also that Euler's formula for a k-component graph, namely $n - e + f = k + 1$, applies also to a k-component graph*, where nodeless edges do not count as components.

If G is a graph*, we denote by $|G|$ the set of points in the edges and vertices of G, ignoring the point at infinity. If G is a graph*, then it makes sense to talk about the faces of G in the open plane—that is, the components of $\mathbb{R}^2 \setminus |G|$. Henceforth, if G is a graph*, when we speak of 'the faces of G', we mean 'the faces of G in the open plane.'

The following basic lemma establishes the importance of piecewise linear and rational piecewise linear graphs.

Theorem 3 *Let A be a connected partition in R; then there exists a finite piecewise linear graph* with no isthmuses whose faces are precisely A. Conversely, let G be a finite piecewise linear graph* with no isthmuses; then the faces of G form a finite connected partition of R.*
The above equivalence also holds if "R" is replaced by "Q" and "piecewise linear" by "rational piecewise linear".

Proof: Suppose that $a_1, \ldots, a_n$ form a connected partition in R. Consider all the half-planes involved in the construction of elements $a_1, \ldots, a_n$. The lines bounding these half-planes form a finite graph* G^* in the obvious way, and the faces of G^* must form a connected partition of basic polygons, say, $b_1, \ldots, b_N$. Moreover, each a_i $(1 \leq i \leq n)$ can certainly be expressed as a sum of various b_j $(1 \leq j \leq N)$. By renumbering if necessary, let $a_1 = b_1 + \ldots + b_m$ for some m $(1 \leq m \leq N)$.
Now remove from G all nodes p such that $p \notin \bigcup\{\mathcal{F}(b_k) | m < k \leq N\}$ and all edges e such that $e \not\subseteq \bigcup\{\mathcal{F}(b_k) | m < k \leq N\}$. The result will be a graph* G_1 in which the faces $b_1, \ldots, b_m$ are merged into a number of faces $f_1, \ldots, f_l$ for some l $(1 \leq l \leq m)$. The union of these faces will then be the set

$$\begin{aligned} b_1 \cup \ldots b_m \cup \{p \in |G| : & p \in \mathcal{F}(b_i) \text{ for some } i (1 \leq i \leq m), \\ & p \notin \mathcal{F}(b_j) \text{ for any } j (m < j \leq N)\} \end{aligned}$$

By lemma 5.3 this set is just $b_1 + \ldots + b_m = a_1$. Since a_1 is connected, $l = 1$ and G_1 contains the face $f_1 = a_1$. Proceeding in the same way for $a_2, \ldots, a_n$ yields a graph* $G = G_n$ with faces $a_1, \ldots, a_n$. That G has no isthmuses follows from the fact that each face of G is regular.
Conversely, suppose that G is a finite piecewise linear graph*; then the edges of G lie on finitely many straight lines $l_1, \ldots, l_n$. Consider the graph G^* made up of all of these lines (extended in both directions). Each face of G^* is a basic polygon; hence each face f_i of G will be divided into a finite number of basic polygons, say, $b_{i,1}, \ldots, b_{i,m_i}$ by a finite number of straight lines. Since G has no isthmuses, f is a regular set, and it is easy to check that no smaller regular open set contains $b_{i,1}, \ldots, b_{i,m_i}$. In other words, $f = b_{i,1} + \ldots + b_{i,m_i} \in R$.
The corresponding proof for Q is identical except for the obvious changes. □

Next we come to some topological results concerning P which, as we shall see in the next section, will have a significant effect on possible alternative models of Th($\mathfrak{P}$).

Lemma 5.5 *There exists a function $e : \mathbb{N} \to \mathbb{N}$ such that, for all $n > 0$, if $a_1, \ldots, a_n$ are disjoint, connected elements of P, then there exist at most $e(n)$ points lying on the boundaries of more than two of these regions.*

Proof: We suppose that p_1, p_2, p_3 are distinct points all lying on the boundaries of a_1, a_2, and a_3 and derive a contradiction. The result then follows by putting $e(n) = n(n-1)(n-2)/3$. Let p_1, p_2, p_3 be as described. Choose points q_1, q_2, q_3 such that $q_i \in a_i$ $(i = 1, 2, 3)$. Since a_1, a_2, and a_3 are polygons, it is clear that for $i = 1, 2, 3$, we can draw three end-cuts in a_i, say $\alpha_{i,1}$, $\alpha_{i,2}$, and $\alpha_{i,3}$ from the point q_i to the points p_1, p_2, and p_3, respectively. (An end-cut in an open set x is a Jordan arc lying in x except for one end-point, which lies on $\mathcal{F}(x)$.) Since we can choose $\alpha_{i,1}$, $\alpha_{i,2}$ and $\alpha_{i,3}$ so that they intersect only at q_i, this gives us a planar embedding of the graph $K_{3,3}$, which is well-known to be non-planar (see, e.g. Bollobás [6], p.19). □

Lemma 5.6 *There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that, for all $n > 0$, if A is any connected partition in P with n members and G is a piecewise linear graph* with no nodes of degree 2 whose faces in the open plane are A, then the size of G is bounded by $f(n)$.*

Proof: Assuming the result that any node of a plane graph of degree greater than 2 must lie on the boundary of at least 3 faces, by lemma 5.5, the number of nodes in G is bounded by a function of n. The lemma then follows from Euler's formula. □

We then have:

Theorem 4 *There exists a function $g : \mathbb{N} \to \mathbb{N}$ such that, for all $n > 0$, there exist at most $g(n)$ n-element connected partitions in P up to homeomorphism.*

Proof: By theorem 3, any such partition is the set of faces of some piecewise linear graph* with no isthmuses, hence of some piecewise linear graph* with no isthmuses and no nodes of degree 2. By lemma 5.6, all such graphs* are of size bounded by $f(n)$. Assuming the result that every abstract graph can be embedded in the closed plane in only finitely many homeomorphically distinct ways, the result follows immediately. □

We note in passing that theorem 4 is false for Euclidean spaces of higher dimension than 2. It is also false for arbitrary partitions of $M(\mathbb{R}^2)$.

The following lemmas are concerned with showing that P is, in a sense that will become clear below, topologically homogeneous.

It is well-known that every finite plane graph G in the open plane can be continuously deformed into piecewise linear plane graph G'. (See, e.g. Bollobás [6], p.16.) Indeed, this can be done in such a way that piecewise linear edges in G are unaffected. In effect, finite plane graphs can have their edges 'straightened out' by a homeomorphism, without affecting any points in those faces whose boundaries involve only straight edges. These results can easily be extended to finite graphs*.

If ν is a homeomorphism of the open plane onto itself and a a subset of the open plane, we write $\nu|_a$ to denote the restriction of ν to a. Then we have:

Lemma 5.7 *Let a, b be connected elements of P such that there is a homeomorphism μ of the open plane onto itself taking a to b. Let $a_1, \ldots, a_n$ be a connected partition of a in P. Then there exists a connected partition $b_1, \ldots, b_n$ of b in R and a homeomorphism ν of the open plane onto itself such that $\nu|_{-a} = \mu|_{-a}$ and $\nu(b_i) = b_i$ for all i $(1 \leq i \leq n)$.*

Proof: We give the proof for $P = R$; the case $P = Q$ is similar.
Let the components of $-a$ be $t_1, \ldots, t_m$. Since $t_1, \ldots, t_m, a_1, \ldots, a_n$ is a connected partition, theorem 3 guarantees that we can find a piecewise linear graph* G with no isthmuses having these elements as faces. Now μ maps a to b, hence the components of $-a$ to the components to $-b$, hence G to a graph* G' with faces $u_1, \ldots, u_m, f_1, \ldots, f_n$, say, where $f_1 + \ldots, f_m = s$. But then we can continuously deform G' to a piecewise linear graph* G'' without affecting any points in $-b$ or its frontier. Hence, the faces of G'' will be $u_1, \ldots, u_m, b_1, \ldots, b_n$, say. Thus, there is a homeomorphism μ' of the closed plane onto itself which is the identity mapping outside b and which maps f_i to b_i, for all i $(1 \leq i \leq n)$. Since G'' clearly contains no isthmuses, theorem 3 guarantees that the faces of G'' will be in S, so that $\nu = \mu' \circ \mu$ is the required homeomorphism. □

Lemma 5.8 *Let a, b be connected elements of P such that there is a homeomorphism μ of the open plane onto itself taking a to b. Let $a' \in P$ satisfying $a' \leq a$. Then there exists $b' \in P$ and a homeomorphism ν of the open plane onto itself such that $\nu|_{-a} = \mu|_{-a}$ and $\nu(a') = b'$.*

Proof: By lemma 5.4, we can find a finite connected partition of a in P some of whose elements sum to a'. The result then follows from lemma 5.7. □

The following notation will prove useful. Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be elements of P. Then we write

$$a_1, \ldots, a_n \sim b_1, \ldots, b_n$$

if there is a homeomorphism μ mapping the open plane on to itself such that $\mu(a_i) = b_i$ for all i $(1 \leq i \leq n)$. Now we can state the lemma guaranteeing homogeneity of P:

Lemma 5.9 *Let $a_1, \ldots, a_n, b_1, \ldots, b_n, a \in P$ such that $a_1, \ldots, a_n \sim b_1, \ldots, b_n$. Then there exists $b \in P$ such that $a_1, \ldots, a_n, a \sim b_1, \ldots b_n, b$.*

Proof: Let μ be a homeomorphism of the closed plane onto itself mapping $a_1, \ldots, a_n$ to $b_1, \ldots, b_n$. Let $c_1, \ldots, c_N$ be all the components of all products of the form $\pm a_1 \cdot \ldots \cdot \pm a_n$ and let $d_1, \ldots d_N$ be all the components of all products of the form $\pm b_1 . \ldots . \pm b_n$. Then, by lemma 5.2, $c_1, \ldots c_N$ and $d_1, \ldots, d_N$ are connected partitions in P, and by renumbering if necessary, μ maps $c_1, \ldots, c_N$ to $d_1, \ldots, d_N$. It suffices to find a $b \in P$ such that $c_1, \ldots, c_N, a \sim d_1, \ldots, d_N, b$.

For all j $(1 \leq j \leq N)$, let $c'_j = a.c_j$. By lemma 5.8, there exists a $d'_j \in P$ and a homeomorphism ν_j mapping c'_j to d'_j and equal to μ outside c_j. Then the mapping

$$\nu = \bigcup \{\nu_j|_{c_j} : 1 \leq j \leq N\} \cup \mu|_{\mathcal{F}(c_1) \cup \ldots \cup \mathcal{F}(c_N)}$$

is a homeomorphism of the open plane onto itself mapping c_j to d_j for all j $(1 \leq j \leq N)$ and mapping $a = c'_1 + \ldots + c'_N$ to $b = d'_1 + \ldots + d'_N \in P$ as required. □

Finally we turn to the embedding of Q in R. Again, we show that this embedding satisfies a topological homogeneity property.

Lemma 5.10 *Let $a_1, \ldots, a_n \in Q$ and $b \in R$. Then there exists $a \in Q$ such that $a_1, \ldots, a_n, b \sim a_1, \ldots, a_n, a$.*

Proof: The result follows from theorem 3 and the fact that any piecewise linear graph can be homeomorphically 'tweaked' so that all line-segments satisfy equations with rational coefficients. The details are routine. □

6 Model-theoretic analysis

This section contains the main technical result of this paper, theorem 8. As we shall see, this theorem has negative consequences for the search for alternative spatial ontologies. Throughout this section, we use the notation $\bar{a}$ to denote an ordered n-tuple $a_1, \ldots, a_n$.

Let us begin by establishing the promised elementary equivalence of $\mathfrak{Q}$ and $\mathfrak{R}$. For the remainder of this paper, we assume that the first order language in question is $\mathcal{L}$ as defined in section 4. First, a reminder from model theory. A *type* $\Gamma(\bar{x})$ in variables $\bar{x}$ is a maximal consistent set of formulae having $\bar{x}$ as their only free variables. Given a model $\mathfrak{A}$, we say that a tuple $\bar{a}$ (of the right arity) belongs to type Γ if $\mathfrak{A} \models \phi[\bar{a}]$ for every $\phi(\bar{x}) \in \Gamma(\bar{x})$.

Lemma 6.1 *Let $\bar{a}$ and $\bar{b}$ be tuples in P such that $\bar{a} \sim \bar{b}$. Then $\bar{a}$ and $\bar{b}$ are of the same type in $\mathfrak{P}$.*

Proof: We prove by induction on the complexity of $\phi(\bar{x})$ that $\mathfrak{P} \models \phi[\bar{a}]$ iff $\mathfrak{P} \models \phi[\bar{b}]$.

If $\phi(\bar{x})$ is $c(t)$, where t is some Boolean combination of the variables $\bar{x}$, then the result is guaranteed by lemma 3.4 and the fact that connectedness is a topological property.

The sole non-trivial recursive case involves showing that, if $\phi(\bar{x})$ is $\exists y \psi(\bar{x}, y)$ and $\mathfrak{P} \models \phi[\bar{a}]$, then $\mathfrak{P} \models \phi[\bar{b}]$. But if $\mathfrak{P} \models \phi[\bar{a}]$, there exists $a \in P$ such that $\mathfrak{P} \models \psi[\bar{a}, a]$. By lemma 5.9, there exists a $b \in P$ such that $\bar{a}, a \sim \bar{b}, b$. By inductive hypothesis, $\mathfrak{P} \models \psi[\bar{b}, b]$. Therefore $\mathfrak{P} \models \phi[\bar{b}]$. □

Lemma 6.1 yields a straightforward result on the relationship between $\mathfrak{R}$ and $\mathfrak{Q}$:

Lemma 6.2 $\mathfrak{Q} \preceq \mathfrak{R}$.

Proof: According to the Tarski-Vaught lemma (Mendelson [28], proposition 2.38), if $\mathfrak{Q} \subseteq \mathfrak{R}$ and, for any n-tuple $\bar{a}$ of $\mathfrak{Q}$ and any formula $\phi(\bar{x})$ of the form $\exists y \psi(\bar{x}, y)$ such that $\mathfrak{R} \models \phi[\bar{a}]$, there exists $b \in Q$ such that $\mathfrak{R} \models \psi[\bar{a}, b]$, then $\mathfrak{Q} \preceq \mathfrak{R}$.

By construction, $\mathfrak{Q} \subseteq \mathfrak{R}$.

Let $\bar{a}$ be an n-tuple of elements of $\mathfrak{Q}$, and let $\phi(\bar{x})$ be any formula of $\mathcal{L}$ of the form $\exists y \psi(\bar{x}, y)$ such that $\mathfrak{R} \models \phi[\bar{a}]$. Then there exists $a \in R$ such that $\mathfrak{R} \models \psi[\bar{a}, a]$. By lemma 5.10, there exists $b \in Q$ such that $\bar{a}, a \sim \bar{a}, b$. By lemma 6.1 applied to $\mathfrak{R}$, $\mathfrak{R} \models \psi[\bar{a}, b]$. □

It follows from lemma 6.2 that $\mathrm{Th}(\mathfrak{Q}) = \mathrm{Th}(\mathfrak{R})$. We have already agreed to denote this set of formulae by $\mathrm{Th}(\mathfrak{P})$.

The language $\mathcal{L}$ allows us to formalize wide a range of topological properties and relationships over the domains $\mathfrak{Q}$ and $\mathfrak{R}$. However, for present purposes, we require only very minimal observations concerning the expressive power of $\mathcal{L}$. Here is one of them: $\mathcal{L}$ contains a formula $\mu_N(\bar{x})$ expressing the notion of being an N-element connected partition.

Lemma 6.3 *For all $N > 0$, let $\mu_N(z_1, \ldots, z_N)$ be the formula*

$$\bigwedge_{1 \leq i \leq N} (c(z_i) \wedge z_i \neq 0) \wedge \bigwedge_{1 \leq i < j \leq N} z_i . z_j = 0 \wedge \sum_{1 \leq i \leq N} z_i = 1 \ .$$

Then for any N-tuple $\bar{c}$ in P, $\mathfrak{P} \models \mu_N[\bar{c}]$ if and only if $\bar{c}$ is a connected partition in P.

Proof: Trivial. □

We will continue to use the abbreviation μ_N in the sequel.

Lemma 6.4 *Let $\bar{a}$ be any n-tuple of elements of P; and let $\bar{x} = x_1, \ldots x_n$ be an n-tuple of variables. Then $\mathfrak{P} \models \psi[\bar{a}]$ for some formula ψ of the form*

$$\exists z_1 \ldots \exists z_N (\mu_N(\bar{z}) \wedge \pi(\bar{x}, \bar{z})) \ ,$$

where $\bar{z} = z_1, \ldots, z_N$ and $\pi(\bar{x}, \bar{z})$ is of the form

$$\bigwedge_{1 \leq i \leq n} (x_i = z_{i,1} + \cdots + z_{i,N_i})$$

such that, for all i $(1 \leq i \leq n)$, $N_i \in \mathbb{N}$, and the $z_{i,j}$ $(1 \leq j \leq N_i)$ are chosen from among the variables $\bar{z}$.

Proof: Immediate from lemmas 5.4 and 6.3. □

(Here and in the sequel, we assume that the notation $\sum_{1 \leq i \leq 0} z_i$ is simply shorthand for the term 0.)

Having set up our mereotopological language and its countable familiar interpretation $\mathfrak{Q}$, the proof that $\mathfrak{Q}$ is minimal proceeds quite simply using standard techniques from model theory. First, we must make more precise the claim that $\mathfrak{Q}$ is minimal. The relevant concept here is that of a *prime* model:

Definition 6.1 *A model $\mathfrak{A}$ is said to be* prime *if, for any model $\mathfrak{B}$, $\mathfrak{A} \equiv \mathfrak{B}$ implies $\mathfrak{A} \preceq \mathfrak{B}$.*

We show that $\mathfrak{Q}$ is prime. It follows that any alternative spatial ontology making the same sentences of our mereotopological language true must contain a copy of $\mathfrak{Q}$, together with some additional elements which make no difference to the formulae satisfied by the elements in that copy of $\mathfrak{Q}$.

The technique we use employs the notion of an *atomic* model:

Definition 6.2 *A formula $\phi(\bar{x})$ is said to be* complete *in a theory T if, for all formulae $\theta(\bar{x})$, exactly one of $T \models \phi \rightarrow \theta$ and $T \models \phi \rightarrow \neg\theta$ hold. A model $\mathfrak{A}$ is said to be* atomic *if any n-tuple $\bar{a}$ in A satisfies a formula $\phi(\bar{x})$ in $\mathfrak{A}$ such that $\phi(\bar{x})$ is complete in* $\mathrm{Th}(\mathfrak{A})$.

Then we have the following standard result from Chang and Keisler [11]:

Theorem 5 (Chang and Keisler: 2.3.4) *A model is countable atomic if and only if it is prime.*

Our task, then, is to show that $\mathfrak{Q}$ is atomic. The following results will also feature in the sequel.

Theorem 6 (Chang and Keisler: 2.3.3) *If $\mathfrak{A}$ and $\mathfrak{B}$ are countable atomic models and $\mathfrak{A} \equiv \mathfrak{B}$, then $\mathfrak{A} \simeq \mathfrak{B}$.*

Theorem 7 (Chang and Keisler: 2.3.13) *Let T be a complete theory. Then T is ω-categorical iff, for each n, T has only finitely many types in $x_1, \ldots, x_n$.*

The following lemma contains the main idea of the proof of theorem 8.

Lemma 6.5 *Every finite connected partition $\bar{c}$ in Q satisfies a complete formula in* $\mathrm{Th}(\mathfrak{Q})$.

Proof: Let $\bar{c}$ have N elements, and let the formula $\mu_N(\bar{z})$ be as defined in lemma 6.3.

Any N-tuple in P satisfying $\mu_N(\bar{z})$ is an N-element connected partition of Q. By theorem 4, there are only finitely many N-element connected partitions of Q up to homeomorphism. Moreover, by lemma 6.1, any two homeomorphic N-tuples belong to the same type $\Gamma(\bar{z})$. Hence, any N-tuple satisfying $\mu_N(\bar{z})$ must belong to one of a finite number of types, $\Gamma_1(\bar{z}), \ldots, \Gamma_k(\bar{z})$ in the variables $\bar{z}$.

Now, $\bar{c}$ satisfies μ_N. Suppose then, without loss of generality, that $\bar{c}$ belongs to type Γ_1. Choose any formula γ_1 consistent with Γ_1 but not with Γ_i for all i $(1 < i \leq k)$. Then $\bar{a}$ satisfies $\mu_N \wedge \gamma_1$. Moreover, all N-tuples satisfying $\mu_N \wedge \gamma_1$ belong to type Γ_1. Hence, if $\theta(\bar{z})$ is any formula in the variables $\bar{z}$, exactly one of $\mathfrak{Q} \models (\mu_N \wedge \gamma_1) \rightarrow \theta$ or $\mathfrak{Q} \models (\mu_N \wedge \gamma_1) \rightarrow \neg\theta$ holds, depending on whether $\theta \in \Gamma_1$. Hence, exactly one of $\mathrm{Th}(\mathfrak{Q}) \models (\mu_N \wedge \gamma_1) \rightarrow \theta$ or $\mathrm{Th}(\mathfrak{Q}) \models (\mu_N \wedge \gamma_1) \rightarrow \neg\theta$ holds. Hence $\mu_N \wedge \gamma_1$ is complete in $\mathrm{Th}(\mathfrak{Q})$. □

Our next task is to extend the result of lemma 6.5 to all tuples of $\mathfrak{Q}$. The process is routine. We first note:

Lemma 6.6 *Let $\bar{x} = x_1, \ldots, x_n$ be an n-tuple of variables; let $\bar{z}$ be an N-tuple of variables; and let $\pi(\bar{x}, \bar{z})$ be of the form defined in lemma 6.4, namely,*

$$\bigwedge_{1 \leq i \leq n} (x_i = z_{i,1} + \cdots + z_{i,N_i})$$

where, for all i $(1 \leq i \leq n)$, $N_i \in \mathbb{N}$, and the $z_{i,j}$ $(1 \leq j \leq N_i)$ are chosen from among the variables $\bar{z}$. Let $\theta(\bar{x})$ be a formula and let $\theta^*(\bar{z})$ be

$$\exists x_1 \ldots \exists x_n (\theta(\bar{x}) \wedge \pi(\bar{x}, \bar{z})) \ .$$

Let $\bar{a}$ be an n-tuple and $\bar{c}$ be an N-tuple such that $\mathfrak{Q} \models \pi[\bar{a}, \bar{c}]$. Then $\mathfrak{Q} \models \theta[\bar{a}]$ iff $\mathfrak{Q} \models \theta^*[\bar{c}]$.

Proof: Trivial. □

Now we come to our main result.

Theorem 8 *$\mathfrak{Q}$ is an atomic model.*

Proof: By lemma 6.4, every n-tuple $\bar{a}$ in Q satisfies a formula of the form:

$$\exists z_1 \ldots \exists z_N (\mu_N(\bar{z}) \wedge \pi(\bar{x}, \bar{z})) \ .$$

So let $\bar{c}$ be an N-tuple such that $\bar{a}, \bar{c}$ satisfies π and $\bar{c}$ satisfies μ_N. Thus, $\bar{c}$ form a finite connected partition in Q. By lemma 6.5, let γ be a complete formula in $\mathrm{Th}(\mathfrak{Q})$ satisfied by $\bar{c}$. Now let $\phi(\bar{x})$ be the formula:

$$\exists z_1 \ldots \exists z_N (\gamma(\bar{z}) \wedge \pi(\bar{x}, \bar{z})) \ .$$

Then $\bar{a}$ satisfies ϕ. We show that ϕ is complete. It suffices to prove that, if $\theta(\bar{x})$ is any formula whose only free variables are $\bar{x}$, and $\bar{b}$ is a tuple in Q satisfying ϕ, then $\mathfrak{Q} \models \theta[\bar{b}]$ if and only if $\mathfrak{Q} \models \theta[\bar{a}]$.

Let θ, $\bar{b}$ be as described. By the construction of ϕ, there exists an N-tuple $\bar{d}$ in Q satisfying γ such that $\bar{b}, \bar{d}$ satisfies π. Now let θ^* be as defined in lemma 6.6. Since both $\bar{a}, \bar{c}$ and $\bar{b}, \bar{d}$ satisfy π, we have:

$$\begin{array}{lcl} \mathfrak{Q} \models \theta[\bar{a}] & \text{if and only if} & \mathfrak{Q} \models \theta^*[\bar{c}] \\ \mathfrak{Q} \models \theta[\bar{b}] & \text{if and only if} & \mathfrak{Q} \models \theta^*[\bar{d}] \; . \end{array}$$

Since $\bar{c}$ and $\bar{d}$ both satisfy γ, and γ is complete, we have:

$$\mathfrak{Q} \models \theta^*[\bar{c}] \quad \text{if and only if} \quad \mathfrak{Q} \models \theta^*[\bar{d}] \; .$$

Putting these results together gives us

$$\mathfrak{Q} \models \theta[\bar{a}] \quad \text{if and only if} \quad \mathfrak{Q} \models \theta[\bar{b}] \; .$$

□

Hence, the familiar model constitutes a minimal ontology for practical mereotopology in the following sense:

Corollary 1 *If* $\mathfrak{A} \models \mathrm{Th}(\mathfrak{P})$, *then* $\mathfrak{Q} \preceq \mathfrak{A}$.

A word about generality is in order at this point. Clearly, theorem 8 applies not just to our language $\mathcal{L}$, but to more or less any language which is "topological" in the sense of lemma 6.1 and which is capable of expressing the notion of a connected partition. Some readers may also have been wondering what might have happened if, instead of using *rational* lines to generate our countable model, we had used lines whose equations had coefficients in some other countable subfield of $\mathbb{R}$, such as the algebraic numbers, for instance. The answer is that, since polygons based on the 'algebraic' lines will satisfy the homogeneity properties of lemmas 5.9 and 5.10, we would be able to prove that the resulting model is atomic and, moreover, is elementarily embedded in $\mathfrak{R}$. But atomic models of the same theory are unique up to isomorphism by theorem 6. Hence, if we had based our countable model on the algebraic rather than the rational numbers, the result would have been essentially the same model. Similar remarks apply to the possibility of liberalizing R and Q by allowing some well-behaved curves other than straight-lines in boundaries—some countable set of circular arcs for example. It is not difficult to show that allowing both straight lines and circular-arcs in boundaries would produce a model elementarily equivalent to $\mathfrak{P}$. If these circular arcs are restricted to having rational centres and radii, this model will be countable, and hence isomorphic to $\mathfrak{Q}$.

The question of course arises as to whether the familiar model $\mathfrak{Q}$ is *strictly* minimal among countable models of $\mathrm{Th}(\mathfrak{P})$, in that there are countable models of $\mathrm{Th}(\mathfrak{P})$ not isomorphic to $\mathfrak{Q}$. The answer is: yes and no.

Theorem 9 $\mathrm{Th}(\mathfrak{P})$ *is not* ω*-categorical.*

Proof: By theorem 7, it suffices to prove that $\mathrm{Th}(\mathfrak{P})$ has countably many types in the single variable x. It is easy to see that, for every positive integer m, the formula $\phi_m(x)$

$$\exists z_1 \ldots \exists z_m \left(\bigwedge_{1 \le i \le m} (c(z_i) \wedge z_i \neq 0) \wedge \bigwedge_{1 \le i < j \le m} \neg c(z_i + z_j) \wedge x = \sum_{1 \le i \le m} z_i \right)$$

is satisfied by all and only those elements in Q having exactly m components. Hence, the $\phi_m(x)$ are all satisfied in $\mathfrak{Q}$; so each can be extended to a type $\Gamma_m(x)$ of $\mathrm{Th}(\mathfrak{P})$. But the $\phi_m(x)$ are also pairwise mutually exclusive in $\mathrm{Th}(\mathfrak{P})$; so no two of them can be extended to the same type in $\mathrm{Th}(\mathfrak{P})$. Hence, $\mathrm{Th}(\mathfrak{P})$ has countably many types in x. □

Thus, there exist countable models of $\mathrm{Th}(\mathfrak{P})$ non-isomorphic to $\mathfrak{Q}$. By theorem 6, these models cannot be atomic, and so cannot be prime. Thus $\mathfrak{Q}$ is, in a strong sense, strictly minimal.

However, it turns out that $\mathrm{Th}(\mathfrak{P})$ satisfies a weakened form of of ω-categoricity. The next theorem shows that the only alternative models to $\mathfrak{P}$ are those containing regions comprising, as we might put it, infinitely many pieces.

Theorem 10 *Any two countable models of* $\mathrm{Th}(\mathfrak{P})$ *omitting the set of formulae*

$$\Sigma(x) = \left\{ \neg\exists z_1, \ldots, \exists z_N \left(\bigwedge_{1 \le i \le N} c(z_i) \wedge x = \sum_{1 \le i \le N} z_i \right) \middle| N \in \mathbb{N} \right\}$$

are isomorphic.

Proof: Let $\mathfrak{A}$ be countable such that $\mathfrak{A} \models \mathrm{Th}(\mathfrak{P})$ and $\mathfrak{A}$ omits $\Sigma(x)$. We show that $\mathfrak{A} \simeq \mathfrak{Q}$.

We show that, since $\mathfrak{A}$ omits $\Sigma(x)$, for every n-tuple $\bar{a}$ in A, there exists an N-tuple $\bar{c}$ satisfying μ_N in A such that the elements of $\bar{a}$ are expressible according to $\mathfrak{A}$ as sums of various elements of $\bar{c}$. That is, lemma 6.4 holds with $\mathfrak{A}$ substituted for $\mathfrak{Q}$.

To see this, simply take all non-zero atoms of the form $\pm a_1 . \cdots . \pm a_n$ and, using the fact that $\mathfrak{A}$ omits $\Sigma(x)$, express each such atom as a sum of elements of A, each of which satisfies $c(x)$ in $\mathfrak{A}$. By lemma 3.3,

$$\mathrm{Th}(\mathfrak{P}) \models \forall x \forall y((c(x) \wedge c(y) \wedge x.y \neq 0) \rightarrow c(x+y)).$$

So we may sum together any non-disjoint pairs of these elements until we have elements $c_1, \ldots, c_N$ satisfying μ_N in $\mathfrak{A}$. Since the atoms $\pm a_1 . \cdots . \pm a_n$ are expressible as the sums of the $c_1, \ldots, c_N$, so are the $a_1, \ldots, a_n$, as required.

Now, using the notation of lemma 6.5, let γ_i be complete formulae in the type Γ_i, $(1 \le i \le k)$ but inconsistent with Γ_j for $j \neq i$. Then any N-tuple in Q satisfying μ_N in $\mathfrak{Q}$ also satisfies one of the γ_i in $\mathfrak{Q}$. Hence, $\mathrm{Th}(\mathfrak{P}) \models \mu_N \rightarrow (\gamma_1 \vee \ldots \vee \gamma_k)$. It follows that any N-tuple in A satisfying μ_N in $\mathfrak{A}$ also satisfies one of the γ_i in $\mathfrak{A}$. Assume without loss of generality that $\mathfrak{A} \models \mu_N[\bar{c}] \wedge \gamma_1[\bar{c}]$. It follows that $\mu_N \wedge \gamma_1$ is a complete formula in $\mathrm{Th}(\mathfrak{P}) = \mathrm{Th}(\mathfrak{A})$ satisfied by $\bar{c}$.

Lemma 6.6 certainly holds with $\mathfrak{A}$ substituted for $\mathfrak{Q}$. Hence the proof of theorem 8 goes through with $\mathfrak{A}$ substituted for $\mathfrak{Q}$, and $\mathfrak{A}$ is atomic. By theorem 6, $\mathfrak{A} \simeq \mathfrak{Q}$. □

7 Related work

The results presented here not only have ramifications for mereotopological theories [10, 9, 37, 18, 2, 7], but they have connections with more practical disciplines. Various logicians have sought to give deductive theories of space and space-time [8, 4, 20, 17, 23], many in terms of modal logics [31, 32, 39, 33, 35, 3, 27]. Recent interest in the analysis of visual languages, such as maps and diagrams [29, 26, 19, 24], has led to the exploration of planar mereotopology in relation to qualitative spatial reasoning, since it is theorized that an important aspect of the semantics of such representations may be given by analysis of spatial relations between representational tokens in the plane. Another more practical area in which ontological issues about the plane are raised is in the construction of computational spatial representations for robots, and in Geographical Information Systems eg: [14, 38, 25]. As we have mentioned, GISs use planar polygonal regions to represent geographic objects. In mobile robotics too, it is common to represent a robot's information about its environment as a planar arrangement of places together with their connection relations, eg:[22, 15, 16, 34].

8 Conclusion

In this paper, we have investigated the possibility of alternative spatial ontologies for "practical" mereotopological reasoning. In order to constrain the problem, we insisted that any such ontology provide a model elementarily equivalent to the 'familiar' polygonal model $\mathfrak{P}$. Our motivation for taking $\mathfrak{P}$ as our point of departure was that many computer packages designed to manipulate spatial data, such as GISs, restrict themselves to piecewise linear objects, without any apparent loss of useful representational power.

We identified rational and real 'versions' of $\mathfrak{P}$, namely $\mathfrak{Q}$ and $\mathfrak{R}$, with the former being countable. The main technical results of this paper state that, although $\mathfrak{Q}$ is not the only countable countable model of $\mathrm{Th}(\mathfrak{P})$, it is, in the sense of elementary embedding, the minimal such model. Thus, the countable alternatives to $\mathfrak{Q}$ all contain a copy of $\mathfrak{Q}$—the 'familiar' regions, plus some 'non-familiar' regions which make no difference to any properties of the familiar regions expressible in $\mathcal{L}$. Thus, in a strong sense, they are less parsimonious. Moreover, we found a simple condition on models of $\mathrm{Th}(\mathfrak{P})$ which determines $\mathfrak{Q}$ up to isomorphism, and provides a useful characterization of the other models of $\mathrm{Th}(\mathfrak{P})$. Apparently, revisions to our ontology of the plane which do not violate the facts of polygonal mereotopology—to the extent they exist at all—must be less parsimonious than the one we started with.

The corresponding problem for the 3-dimensional case, or for domains in which lemma 5.1 fails, is open. A complete axiomatization of $\mathrm{Th}(\mathfrak{P})$ has been developed in Pratt and Schoop [30].

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