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Abstract

In recent years, there has been renewed interest in the development of formal languages for describing mereological (part-whole) and topological relationships between objects in space. Typically, the non-logical primitives of these languages are properties and relations such as 'x is connected' or 'x is a part of y', and the entities over which their variables range are, accordingly, not *points*, but *regions*: spatial entities other than regions are admitted, if at all, only as logical constructs of regions. This paper considers two first-order mereotopological languages, and investigates their expressive power. It turns out that these languages, notwithstanding the simplicity of their primitives, are surprisingly expressive. In particular, it is shown that infinitary versions of these languages are adequate to express (in a sense made precise below) all topological relations over the domain of polygons in the closed plane.

1 Introduction

In recent years, there has been renewed interest in the development of formal languages for reporting mereological (part-whole) and topological information. Typically, the non-logical primitives of these languages are predicates with intended readings such as 'x is connected' or 'x is a part of y', and the entities over which their variables range are, accordingly, not points, but regions. These languages originate in that branch of philosophical logic which attempts to reconstruct topology and geometry using an ontology in which the primitive entities are extended regions (volumes [21], lumps [19], spheres [29]; see also [7, 8, 11, 13]). However, the motivation for much of the recent interest, particularly in the AI community, is more computational than metaphysical. The hope is that such region-based descriptions of space, by limiting the expressive resources at the command of their users, may make the representation of spatial information more efficient for the purposes of commonsense reasoning.

Consider, for example, the shapes depicted in figure 1. None is a very likely candidate for the region of space occupied by any of the two-dimensional abstractions of everyday discourse: plots of land on a map, available floorspace in a warehouse, flower beds in a garden. So, bearing in mind the general trade-off in AI between expressive power and ease of computation, perhaps we should opt for a spatial ontology from which such pathological examples are excluded. And, if we employ a region-based ontology and a language to describe it whose primitives are interpreted directly as relations between regions, then perhaps—so the supporters of region-based ontologies hope—we can be more selective about the kinds of regions we admit.

Whether these considerations really justify the use of a mereotopological language interpreted over a region-based ontology is not an issue we shall discuss in this paper. (For an alternative, non-region-based approach to well-behaved regions, see e.g. van den Dries [30].) However, if we do adopt such a language, it is important to establish what this language lets us say. That is, we must determine its expressive power. Relatively little effort has been devoted to this question in the mereotopological literature, though one exception is Gotts' [14] attempt to define a torus within an ontology of three-dimensional regions. The issue of how much expressive power we want for a mereotopological language for commonsense qualitative spatial reasoning is a delicate one: too little, and we lose the ability to report information of practical importance; too much, and we re-instate the useless distinctions such languages are supposed to avoid. The present paper undertakes a systematic analysis of the expressivity of two first-order mereotopological languages interpreted over well-behaved regions in the open and closed real planes. These two languages differ in their choice of primitive predicates. The first employs the predicates $x \leq y$ ('x is a part of y') and $c(x)$ ('x is connected') [5, 25]. The second employs the single primitive $C(x, y)$ ('x and y are in contact') [1, 9, 15, 26, 31]. As we shall see, these languages, notwithstanding the simplicity of their primitives, turn out to be surprisingly expressive. As to whether they are *too* expressive for the purposes of AI, we express no opinion here; however, we do claim at least to have answered the relevant mathematical questions.

The plan of the paper is as follows. The next section introduces the mereotopological languages under consideration. Section 3 presents the main results. Section 4 presents some basic results about the spatial ontologies considered in this paper. Section 5 contains the proofs of the new theorems reported here.

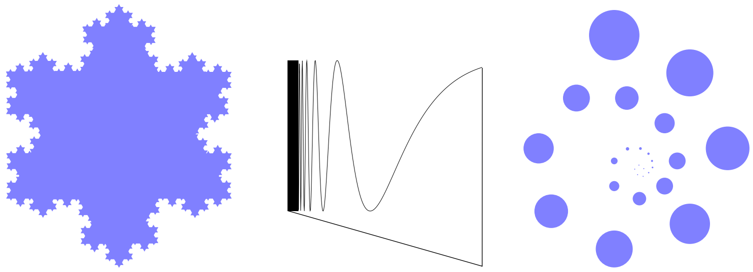

Figure 1: Examples of pathological regions: a) fractal; b) infinitely oscillating boundary; c) infinitely many components

2 Mereotopologies

Let X be a topological space. By a *mereotopology* over X, we understand any structure $\mathfrak{A}$, in the sense of model theory, with domain $A \subseteq \wp(X)$, in which all the non-logical primitives receive standard mereological and topological interpretations. We give a formal definition below. Specifically, we always interpret the predicates c, $\leq$ and C as follows.

$$\begin{array}{lcl} [\![c(x)]\!]^{\mathfrak{A}} & = & \{a \in A | a \text{ is connected}\} \\ [\![x \leq y]\!]^{\mathfrak{A}} & = & \{(a,b) \in A^2 | a \subseteq b\} \\ [\![C(x,y)]\!]^{\mathfrak{A}} & = & \{(a,b) \in A^2 | [a] \cap [b] \neq \emptyset\}. \end{array}$$

(Here we take $[x]$ to denote the closure of x, reserving the more usual notation $\bar{x}$ for n-tuples.) We adopt the convention that $\mathcal{L}(\sigma)$ denotes the first-order language with signature σ, and $\mathfrak{A}(\sigma)$ denotes a structure interpreting $\mathcal{L}(\sigma)$ over domain A. This paper is chiefly concerned with the first-order languages $\mathcal{L}(c, \leq)$ and $\mathcal{L}(C)$, and domains of particularly well-behaved regions in the open and closed real planes.

Given a mereotopology $\mathfrak{A}(\sigma)$, the set of $\mathcal{L}(\sigma)$-sentences true in $\mathfrak{A}(\sigma)$ is denoted $Th(\mathfrak{A}(\sigma))$; and various questions naturally arise about $Th(\mathfrak{A}(\sigma))$ and its relation to A. These questions fall into three salient classes. First, we have the ontological questions. What models does $Th(\mathfrak{A}(\sigma))$ have? Is $Th(\mathfrak{A}(\sigma))$ perhaps ω-categorical? If not, can we identify those countable models which are, in some sense, *simplest*? Of particular interest are the models of $Th(\mathfrak{A}(\sigma))$ where the domain is some subset of $\wp(X)$ other than A: what collections of regions $A' \subseteq \wp(X)$ yield models with the same theory as A? Second, we have the purely computational/syntactic questions. Can $Th(\mathfrak{A}(\sigma))$ be characterized axiomatically? Is $Th(\mathfrak{A}(\sigma))$ decidable? Third, we have questions about the expressivity of the language $\mathcal{L}(\sigma)$ under the interpretation $\mathfrak{A}(\sigma)$. What relations over A can be defined by formulas of $\mathcal{L}(\sigma)$? Can any signatures σ be identified as *topologically adequate* in that they allow us to express all possible topological relations over A?

These questions are especially important when X is a familiar topological space (for example, $\mathbb{R}^2$ or $\mathbb{R}^3$) and A is a natural class of well-behaved regions. For in that case, the set of sentences $Th(\mathfrak{A}(\sigma))$ represent, in some sense, the facts of 'commonsense' mereology and topology, and its models the possible spatial ontologies needed to sustain them. We turn now to the issue of which regions we might consider as well-behaved for these purposes.

As one of the main motivations for studying mereotopologies is the desire to avoid distinctions deemed useless for commonsense reasoning, most approaches restrict attention to so-called regular sets as a way of finessing the issue of whether regions include their boundary points. If X is a topological space, we say that $x \subseteq X$ is *regular open* if $x = [x]^\circ$, where $[x]$ denotes the closure of x and x° denotes the interior of x. It is easy to see that no two regular open sets differ only with respect to boundary points, and that, for every $x \subseteq X$, there is a regular set y differing from x only with respect to boundary points. Moreover, it is well known that the set of regular open sets of a topological space X, denoted $RO(X)$, forms a Boolean algebra $(RO(X), +, \cdot, -, X, \emptyset)$ where $x+y := [x \cup y]^\circ$, $x \cdot y := x \cap y$ and $-x := X \setminus [x]$ (see e.g. [17], theorem 1.37), so the resulting ontology is mathematically tractable, and allows the definition of intuitively appealing mereological operations.

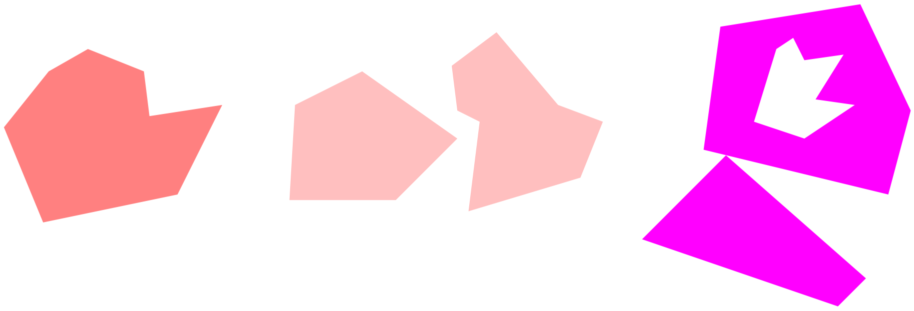

Figure 2: Three polygonal regions

Regular sets may succeed in avoiding the distinction between closed and open sets; but they do not do a satisfying job of restricting attention to regions of space that could be occupied by any of the two-dimensional abstractions of everyday discourse. For example, the open set in figure 1b) defined by

$$\{(x, y)|0 < x < 1, \ -1 - x < y < \sin(1/x)\}$$

is regular, but it is hardly the sort of thing that is useful in everyday situations. In this paper, we consider the case where our domain of regions A is the set of *regular open polygons* in the open or closed planes. This ontology is, admittedly, rather more spartan than one might perhaps wish for. Nevertheless, it has the virtues of being both mathematically tractable and—given the almost universal use of polygonal approximations in electronic representations of spatial data—of indisputable practical interest. Possible liberalizations of this ontology are considered in the final section. Our first task is to define exactly what we mean by a regular open polygon.

Any line in the open plane cuts it into two connected, open sets, called *half-planes*. Let us call the intersection of finitely many half-planes a *basic polygon*. A *regular open polygon* (hereinafter, simply: *polygon*) is then defined as the sum, in the Boolean algebra of regular open sets of $\mathbb{R}^2$, of any finite set of basic polygons. We denote the set of all polygons in the open plane by R. Figure 2 shows three polygons. Note that polygons need not be connected, may contain 'holes', and may be unbounded. Regarding the closed plane $Z^2 = \mathbb{R}^2 \cup \{\infty\}$, i.e. the one-point-compactification of the open plane, it is obvious that half-planes, basic polygons and polygons can be defined in the same way as for the open plane. We denote the set of polygons in the closed plane by $\bar{R}$. We use the symbol $\tilde{R}$ to stand ambiguously for either R or $\bar{R}$. Similarly, if σ is the signature of a mereotopological language, we write $\tilde{\mathfrak{R}}(\sigma)$ to stand ambiguously for either $\mathfrak{R}(\sigma)$ or $\bar{\mathfrak{R}}(\sigma)$.

The present paper addresses questions concerning the expressivity of $\mathcal{L}(c, \leq)$ and $\mathcal{L}(C)$ in the models $\tilde{\mathfrak{R}}(c, \leq)$ and $\tilde{\mathfrak{R}}(C)$ respectively. Questions concerning the models of $Th(\mathfrak{R}(c, \leq))$ are addressed in [25]; questions concerning the syntactic characterization of $Th(\mathfrak{R}(c, \leq))$ are addressed in [24]. Some of the theorems reported below allow these earlier results to be extended in various ways, but we shall not pause to make these extensions explicit.

3 The Main Results

The following definitions are needed to state all the main results of this paper.

Definition 3.1 *Let A be a set. By a* relation *over A we mean simply a subset $\mathcal{C}$ of A^n for some $n \geq 1$. If $\mathfrak{A}(\sigma)$ is a structure interpreting $\mathcal{L}(\sigma)$ over A, we say that a relation $\mathcal{C}$ is $\mathcal{L}(\sigma)$-*definable in $\mathfrak{A}(\sigma)$ *if, for some formula $\phi(x_1, \ldots, x_n) \in \mathcal{L}(\sigma)$ with $n \geq 1$, $\mathcal{C} = \{\bar{a} \in A^n | \mathfrak{A}(\sigma) \models \phi[\bar{a}]\}$.*

Our first results address the relative expressivity of $\mathcal{L}(C)$ and $\mathcal{L}(c, \leq)$. It transpires:

Theorem 3.2 *All relations over R which are $\mathcal{L}(c, \leq)$-definable in $\mathfrak{R}(c, \leq)$ are also $\mathcal{L}(C)$-definable in $\mathfrak{R}(C)$, but not vice-versa.*

Theorem 3.3 *All relations over $\bar{R}$ which are $\mathcal{L}(c, \leq)$-definable in $\bar{\mathfrak{R}}(c, \leq)$ are also $\mathcal{L}(C)$-definable in $\bar{\mathfrak{R}}(C)$, and vice-versa.*

Thus, in the open plane, $\mathcal{L}(C)$ is strictly more expressive than $\mathcal{L}(c, \leq)$; but this advantage is lost when we move to the closed plane. In consequence, for the purposes of discussing expressivity in the closed plane, we can speak of the language $\mathcal{L}$ to refer indifferently to either of $\mathcal{L}(c, \leq)$ or $\mathcal{L}(C)$, and of the mereotopology $\bar{\mathfrak{R}}$ to refer indifferently to either of $\bar{\mathfrak{R}}(c, \leq)$ or $\bar{\mathfrak{R}}(C)$.

Of course, expressivity of one mereotopological language relative to another is all well and good, but what we really want to know is how expressive these languages are relative to familiar notions from topology. In order to articulate this question, we need some more definitions.

Definition 3.4 *Let X be a topological space and $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ subsets of X. We say that $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are* similarly situated, *and write $a_1, \ldots, a_n \sim b_1, \ldots, b_n$, if there is a homeomorphism of the space X onto itself taking a_i to b_i for all i ($1 \leq i \leq n$). A relation $\mathcal{C}$ over a subset A of $\wp(X)$ is called* topological *if for all $\bar{a} \in \mathcal{C}$ and $\bar{b}$ in A^n (with n the length of $\bar{a}$) $\bar{a} \sim \bar{b}$ implies $\bar{b} \in \mathcal{C}$. A subset A of $\wp(X)$ is said to be* topologically homogeneous *if, whenever $\bar{a}, \bar{b}$ are n-tuples from A such that $\bar{a} \sim \bar{b}$ and a is an element of A, there exists $b \in A$ such that $\bar{a}, a \sim \bar{b}, b$.*

Thus, a topologically homogeneous set $A \subseteq \wp(X)$ need not be closed under arbitrary homeomorphisms of X; however, any two similarly situated n-tuples in A must 'look alike' in terms of their topological relations to other elements of A. Anticipating lemma 4.5 below, we note that the open- and closed-plane polygonal domains R and $\bar{R}$ are both topologically homogeneous.

Definition 3.5 *Let X be a topological space and $\mathfrak{A}(\sigma)$ be a structure with domain $A \subseteq \wp(X)$. $\mathfrak{A}(\sigma)$ is said to be a* mereotopology *if A is topologically homogeneous and every atomic $\mathcal{L}(\sigma)$-formula defines a topological relation over A. A formula in $\mathcal{L}(\sigma)$ is said to be* topologically complete *in the mereotopology $\mathfrak{A}(\sigma)$ if, for all n-tuples $\bar{a}$ and $\bar{b}$ in A, $\mathfrak{A}(\sigma) \models \phi[\bar{a}]$ and $\mathfrak{A}(\sigma) \models \phi[\bar{b}]$ implies $\bar{a} \sim \bar{b}$.*

Now we can characterize the expressivity of the mereotopological languages considered above from a topological point of view:

Theorem 3.6 *Let $\mathfrak{A}(\sigma)$ be a mereotopology. All $\mathcal{L}(\sigma)$-definable relations in $\mathfrak{A}(\sigma)$ are topological.*

Theorem 3.7 *In the open-plane polygonal mereotopology $\mathfrak{R}(C)$, every n-tuple satisfies a topologically complete formula of $\mathcal{L}(C)$.*

Theorem 3.8 *In the closed-plane polygonal mereotopology $\bar{\mathfrak{R}}$, every n-tuple satisfies a topologically complete formula of $\mathcal{L}$.*

Theorem 3.6 is easy to prove and sets an obvious upper bound on the expressivity of any mereotopological language: it cannot distinguish between similarly situated n-tuples. Theorems 3.7 and 3.8, by contrast, are considerably harder to prove, and set strong lower bounds on the expressivity of certain specific mereotopological languages. Thus, theorem 3.7 states that, for any n-tuple from R, the language $\mathcal{L}(C)$ is expressive enough to characterize that n-tuple upto the relation of being similarly situated; likewise, theorem 3.8 states that, for any n-tuple from $\bar{R}$, the languages $\mathcal{L}(c, \leq)$ and $\mathcal{L}(C)$ are both expressive enough to characterize that n-tuple upto the relation of being similarly situated. Theorem 3.7 would not hold with the signature $\langle C \rangle$ replaced by $\langle c, \leq \rangle$.

Theorems 3.7 and 3.8 can be used to show that the signature $\langle C \rangle$, and, in the case of the closed plane, the signature $\langle c, \leq \rangle$ as well, are *topologically adequate* over the polygonal domains in the following sense. Suppose we construct the infinitary languages $\mathcal{L}^{\omega_1}(c, \leq)$ and $\mathcal{L}^{\omega_1}(C)$ in exactly the same way as $\mathcal{L}(c, \leq)$ and $\mathcal{L}(C)$, except that, if $\Phi(x_1, \ldots, x_n)$ is a countable set of formulas in the variables $x_1, \ldots, x_n$, then $\bigwedge \Phi(x_1, \ldots, x_n)$ and $\bigvee \Phi(x_1, \ldots, x_n)$ are also formulas. (Thus, formulas of $\mathcal{L}^{\omega_1}(c, \leq)$ and $\mathcal{L}^{\omega_1}(C)$, although infinitary, may contain only finitely many variables.) We have:

Theorem 3.9 *Let $\mathcal{C}$ be a relation over the open-plane polygonal domain R. Then the following are equivalent:*

(i) $\mathcal{C}$ *is topological;*

(ii) $\mathcal{C}$ *is $\mathcal{L}^{\omega_1}(C)$-definable in the structure $\mathfrak{R}(C)$.*

Let $\mathcal{C}$ be a relation over the closed-plane polygonal domain $\bar{R}$. Then the following are equivalent:

(i) $\mathcal{C}$ *is topological;*

(ii) $\mathcal{C}$ *is $\mathcal{L}^{\omega_1}(C)$-definable in the structure $\bar{\mathfrak{R}}(C)$;*

(iii) $\mathcal{C}$ *is $\mathcal{L}^{\omega_1}(c, \leq)$-definable in the structure $\bar{\mathfrak{R}}(c, \leq)$.*

It is obvious, by a simple counting argument, that no such result as theorem 3.9 could hold for the finitary versions of these languages.

4 Basic properties of $\tilde{\mathfrak{R}}$

In this section, we review the properties of $\mathfrak{R}$ and $\bar{\mathfrak{R}}$ on which the above expressivity results depend. The following easy result shows that the polygons form sensible collections of regions in the open- and closed-planes.

Theorem 4.1 *The structure $\mathfrak{R}$ is a Boolean subalgebra of $RO(\mathbb{R}^2)$; the structure $\bar{\mathfrak{R}}$ is a Boolean subalgebra of $RO(Z^2)$.*

Proof: We need only check that these structures are closed under the Boolean operations in their respective regular open algebras. But this is immediate from the distribution laws for $+$ and $\cdot$, and the fact that the complement of a half-plane is a half-plane. □

For $r, s \in \tilde{R}$, we can think of the Boolean sum $r+s$ as formed by taking the union of r and s and 'rubbing out' any line segments lying on the boundaries of both r and s, so as to guarantee that the resulting region is regular. Thus, the operation $+$ corresponds to an intuitively appealing notion of mereological agglomeration. It will be convenient, in the sequel, to denote *disjoint* sums with the symbol $\oplus$. Thus, we write $r = s \oplus t$ if $r = s + t$ and s, t are nonempty and disjoint. Since the operations $+$, $\cdot$, $-$ and $\oplus$, as well as the constants 0 and 1 (denoting the empty set and the whole space, respectively) are clearly $\mathcal{L}(c, \leq)$-definable in $\tilde{\mathfrak{R}}(c, \leq)$, we may henceforth use these symbols in $\mathcal{L}(c, \leq)$-formulas, as a shorthand for their definitions.

The well-behavedness of polygons manifests itself in the following important properties. (We omit the proof of the first one.)

Lemma 4.2 *Every component of $r \in \tilde{R}$ is an element of $\tilde{R}$.*

Here and in the sequel, we take $\partial(r)$ to stand for the boundary of r: i.e. $\partial(r) = [r] \setminus r$.

Definition 4.3 *Let x be a subset of the open or closed plane. Let $q \in x$ and $p \in \partial(x)$. An* end-cut *from q to p in x is a Jordan arc α from q to p lying entirely in x except for the point p. In such a case, we say that p is accessible from q by α (in x).*

Lemma 4.4 *Let $r \in \tilde{R}$ and $p \in \partial(r)$. Then p is accessible by means of a linear endcut from some point in r.*

Proof: If r is a basic polygon, the result is immediate. If $r = r_1 + \ldots + r_n$, where the r_i are basic polygons, then $p \in \partial(r)$ implies $p \in \partial(r_i)$ for some i, so is accessible by means of a linear endcut from some point in $r_i \subseteq r$. □

It follows, of course, that if r is connected, then p is accessible by means of a piecewise linear endcut from every point in r. This result does not hold for all regular open sets of the plane. For example, the wiggly region in figure 1b) is regular, and indeed connected, but contains boundary points not accessible (even by an arbitrary Jordan arc) from its interior.

The following lemma states that, in the open and closed planes, the set of polygons forms a topologically homogeneous set of regions. (Hence, it is permissible to speak of the *mereotopologies* $\tilde{\mathfrak{R}}(c,\leq)$ and $\tilde{\mathfrak{R}}(C)$.)

Lemma 4.5 *$\tilde{R}$ is topologically homogeneous: if $\bar{a},\bar{b}$ are n-tuples of $\tilde{R}$ such that $\bar{a}\sim\bar{b}$, and $a\in\tilde{R}$, then there exists $b\in\tilde{R}$ such that $\bar{a},a\sim\bar{b},b$.*

In effect, the lemma is a restatement of the result that any finite graph in the open or closed planes plane can be homeomorphically straightened out so that its edges all lie on finitely many straight line segments. A detailed proof of this lemma in the case $\tilde{R}=R$ is given in Pratt and Lemon [25, lemma 4.13]; the proof in the case $\tilde{R}=\bar{R}$ is virtually identical and need not be repeated here.

The following easy but fundamental lemma will be used throughout this paper. It illustrates the importance of the notion of topological homogeneity.

Lemma 4.6 *Let $\mathfrak{A}(\sigma)$ be a mereotopology. Let $\bar{a}$, $\bar{b}$ be n-tuples in A and $\phi(\bar{x})$ a formula of $\mathcal{L}(\sigma)$. If $\mathfrak{A}\models\phi[\bar{a}]$ and $\bar{a}\sim\bar{b}$, then $\mathfrak{A}\models\phi[\bar{b}]$.*

Proof: By induction on the complexity of ϕ. If ϕ is an atomic formula, then, by assumption, its interpretation is a topological relation. The only non-trivial recursive case is where ϕ is $\exists x\psi(\bar{x},x)$. Suppose, then $\mathfrak{A}\models\phi[\bar{a}]$. Let $a\in A$ be such that $\mathfrak{A}\models\psi[\bar{a},a]$. Since A is topologically homogeneous, let $b\in A$ satisfy $\bar{a},a\sim\bar{b},b$. By inductive hypothesis, $\mathfrak{A}\models\psi[\bar{b},b]$, and hence $\mathfrak{A}\models\phi[\bar{b}]$. □

The next result shows that, although the open and closed planes are non-homeomorphic topological spaces, the mereotopologies $\mathfrak{R}(c,\leq)$ and $\bar{\mathfrak{R}}(c,\leq)$ are indistinguishable. A full proof is given in Pratt and Schoop [24, lemma 2.5].

Lemma 4.7 *The models $\mathfrak{R}(c,\leq)$ and $\bar{\mathfrak{R}}(c,\leq)$ are isomorphic.*

Sketch of proof: Let $f:RO(Z^2)\rightarrow RO(\mathbb{R}^2)$ be defined by $f(r)=r\setminus\{\infty\}$, and let $g:RO(\mathbb{R}^2)\rightarrow RO(Z^2)$ be defined by $g(s)=[s]^{\circ}$ (where the closure- and interior-operations are to be carried out in the closed plane topology). Then it is straightforward to verify that f and g are inverses of each other and that each is a Boolean algebra isomorphism. Since, moreover, f and g map half-planes to themselves, it follows that $f:\bar{R}\rightarrow R$ is also a Boolean algebra isomorphism and the restriction of g to R is its inverse. Finally, if $r\in\bar{R}$, then r is evidently connected in the closed plane topology if and only if $f(r)$ is connected in the open plane topology. □

We note in passing that f and g are not isomorphisms between $\mathfrak{R}(C)$ and $\bar{\mathfrak{R}}(C)$, since the extension of the predicate $C(x,y)$ is clearly not preserved. In fact, as we shall see below (corollary 5.9), $\mathfrak{R}(C)$ and $\bar{\mathfrak{R}}(C)$ are not even elementarily equivalent.

5 The Proofs

5.1 Equivalence of the finitary languages

We turn first to the proof that $\mathcal{L}(C)$ is at least as expressive as $\mathcal{L}(c,\leq)$. Specifically, we prove that the interpretations of the primitives c and $\leq$ in $\mathcal{L}(c,\leq)$ are $\mathcal{L}(C)$-definable in $\tilde{\mathfrak{R}}(C)$.

Lemma 5.1 *Let $r_1,r_2\in\tilde{R}$. Then $r_1\leq r_2$ if and only if $\tilde{\mathfrak{R}}(C)\models\phi_{\leq}[r_1,r_2]$, where $\phi_{\leq}(x,y)$ is the $\mathcal{L}(C)$-formula $\forall z(C(x,z)\rightarrow C(y,z))$.*

Proof: If $r_1 \leq r_2$ then $[r_1] \subseteq [r_2]$, so $[t] \cap [r_1] \neq \emptyset$ implies $[t] \cap [r_2] \neq \emptyset$ for any set t. Conversely, if $r_1 \cdot (-r_2)$ is non-empty, we can certainly find a polygon t lying in its interior, so that $[t] \cap [r_1] \neq \emptyset$, but $[t] \cap [r_2] = \emptyset$. □

Since the subset relation $\leq$ on $\tilde{R}$ is $\mathcal{L}(C)$-definable in $\tilde{\mathfrak{R}}(C)$, we may henceforth write the symbol $\leq$ in $\mathcal{L}(C)$-formulas, as a shorthand for $\phi_{\leq}$. It follows in addition that the Boolean functions $+$, $\cdot$, $-$ and $\oplus$, as well as the constants 0 and 1, are also $\mathcal{L}(C)$-definable, so again, we may henceforth write these symbols in $\mathcal{L}(C)$-formulas as a shorthand for their definitions.

To establish that the property of connectedness is $\mathcal{L}(C)$-definable in $\tilde{\mathfrak{R}}(C)$, we use the following fact:

Lemma 5.2 *Let $r \in \tilde{R}$. Then r is connected if and only if, for all nonempty, disjoint $r_1, r_2 \in \tilde{R}$ such that $r_1 + r_2 = r$, we have $\partial(r_1) \cap \partial(r_2) \cap r \neq \emptyset$.*

Proof: Suppose r is connected and r_1, $r_2 \in \tilde{R}$ satisfy $r_1 \oplus r_2 = r$. Let $q_i \in r_i$ $(i = 1, 2)$ and let α be a Jordan arc in r from q_1 to q_2, which exists by the connectedness of r. Let p be a point on α in $\partial(r_1)$. Then it is easy to see that $p \in \partial(r_1) \cap \partial(r_2) \cap r$. Conversely, suppose that r is not connected. Let r_1 be any component of r. By lemma 4.2 $r_1 \in \tilde{R}$. Let $r_2 = r \cdot -r_1$, so that $r_1 \oplus r_2 = r$. If $p \in \partial(r_1) \cap \partial(r_2)$, then, by lemma 4.4, we can find a point $q_i \in r_i$ and an end-cut α_i in r_i from q_i to p $(i = 1, 2)$. If, in addition, $p \in r$, then α_1 and $-\alpha_2$ form a Jordan arc in r from q_1 to q_2, contradicting the fact that q_1 and q_2 lie in different components of r. Hence $\partial(r_1) \cap \partial(r_2) \cap r = \emptyset$. □

Lemma 5.3 *Let $r \in \tilde{R}$. Then r is connected if and only if $\tilde{\mathfrak{R}}(C) \models \phi_c[r]$, where $\phi_c(x)$ is the $\mathcal{L}(C)$-formula*

$$\forall x_1 \forall x_2 (x_1 \oplus x_2 = x \rightarrow \exists x'_1 \exists x'_2 (x'_1 \leq x_1 \wedge x'_2 \leq x_2 \wedge C(x'_1, x'_2) \wedge \neg C(x'_1 + x'_2, -x))).$$

Proof: It is straightforward to show by lemma 4.4 that for nonempty and disjoint regions r_1 and r_2, $\partial(r_1) \cap \partial(r_2) \cap (r_1 + r_2) \neq \emptyset$ if and only if r_1 and r_2 satisfy the formula

$$\exists x'_1 \exists x'_2 (x'_1 \leq x_1 \wedge x'_2 \leq x_2 \wedge C(x'_1, x'_2) \wedge \neg C(x'_1 + x'_2, -(x_1 + x_2)))$$

in $\tilde{\mathfrak{R}}$. The result then follows from lemma 5.2. Thus, we may henceforth write the symbol c in $\mathcal{L}(C)$-formulas, understanding it as a shorthand for ϕ_c. □

Our next task is to show that $\mathcal{L}(c, \leq)$ is at least as expressive as $\mathcal{L}(C)$ in the closed plane. Specifically, we prove that the interpretation of the primitive C is $\mathcal{L}(c, \leq)$-definable in the model $\bar{\mathfrak{R}}(c, \leq)$. The proof of the next lemma assumes familiarity with the concepts of 1-chains and 1-cycles from topology.

Lemma 5.4 *Let $s_1, s_2, t \in \bar{R}$ with $-(s_1 + t)$, $-(s_2 + t)$ and t all connected, and $[s_1] \cap [s_2] = \emptyset$. Then $-(s_1 + s_2 + t)$ is also connected.*

Proof: Alexander's lemma ([20], theorem V.9.1.2) states the following. Let F_1 and F_2 be closed sets in the closed plane Z^2. If the points p and q bound the 1-chains κ_i not meeting F_i $(i = 1, 2)$ and if the 1-cycle $\kappa_1 + \kappa_2$ bounds in $Z^2 \setminus (F_1 \cap F_2)$, then p and q are not separated by $F_1 \cup F_2$. Assume $-(s_1 + s_2 + t)$ is nonempty, since otherwise the lemma is trivially true. Set $F_i = [s_i + t]$ $(i = 1, 2)$. Choose any distinct $p, q \in -(s_1 + s_2 + t)$, and let κ_i be a 1-chain with endpoints p and q, such that $|\kappa_i| \cap F_i = \emptyset$. (Note that $-(s_i + t) = Z^2 \setminus F_i$ is connected by hypothesis.) It is a standard result ([20], theorem V.6.2) that, if G is any set whose complement is connected, then any 1-cycle bounds in G. Now, since $F_1 \cap F_2 = [s_1 + t] \cap [s_2 + t] =([s_1] \cup [t]) \cap ([s_2] \cup [t]) = ([s_1] \cap [s_2]) \cup [t] = [t]$, and since t (and hence $[t]$) is connected, the 1-cycle $\kappa_1 + \kappa_2$ bounds in $Z^2 \setminus (F_1 \cap F_2)$. It follows by Alexander's lemma that p and q are not separated by $F_1 \cup F_2 = [s_1] \cup [s_2] \cup [t] = [s_1 + s_2 + t]$. In other words, $-(s_1 + s_2 + t)$ is

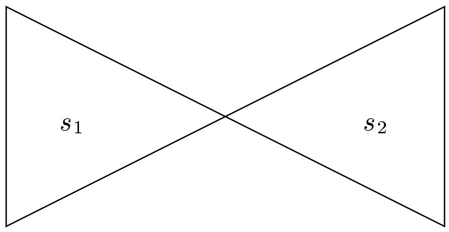

Figure 3: Regions s_1 and s_2 forming a bowtie.

connected. □

In the sequel we write $\pi(y_1, y_2)$ to abbreviate the formula

$$\exists z(c(-(y_1+z)) \wedge c(-(y_2+z)) \wedge c(z) \wedge \neg c(-(y_1+y_2+z))).$$

Lemma 5.5 *Let $r_1, r_2 \in \bar{R}$. Then $[r_1] \cap [r_2] \neq \emptyset$ if and only if $\bar{\mathfrak{R}}(c, \leq) \models \phi_C[r_1, r_2]$, where $\phi_C(x_1, x_2)$ is the $\mathcal{L}(c, \leq)$-formula:*

$$\exists y_1 \exists y_2 (y_1 \leq x_1 \wedge y_2 \leq x_2 \wedge \pi(y_1, y_2)).$$

Proof: The if-direction is immediate given lemma 5.4. For the only-if direction, let us say that s_1 and s_2 *form a bowtie* if they are similarly situated to the arrangement shown in figure 3. Visibly, $\bar{\mathfrak{R}}(c, \leq) \models \pi[s_1, s_2]$. It is easy to see by lemma 4.4 that if $[r_1] \cap [r_2] \neq \emptyset$ then we can find $s_1 \leq r_1$, $s_2 \leq r_2$ such that s_1 and s_2 form a bowtie, whence $\bar{\mathfrak{R}}(c, \leq) \models \phi_C[r_1, r_2]$. □

Thus, the relation $[r_1] \cap [r_2] \neq \emptyset$ is $\mathcal{L}(c, \leq)$-definable in $\bar{\mathfrak{R}}(c, \leq)$.

Finally, we show that there is a relation over R which is $\mathcal{L}(C)$-definable but not $\mathcal{L}(c, \leq)$-definable. The relation of *boundedness* will do nicely.

Lemma 5.6 *Let $s_1, s_2, t \in R$ with $-(s_1+t)$, $-(s_2+t)$ and t all connected, $-(s_1+s_2+t)$ not connected, and $[s_1] \cap [s_2] = \emptyset$. Then s_1 and s_2 are unbounded.*

Proof: Let $g : \mathfrak{R}(c, \leq) \to \bar{\mathfrak{R}}(c, \leq)$ be the isomorphism defined in lemma 4.7. Then $g(-(s_1+t)) = -(g(s_1)+g(t))$, $g(-(s_2+t)) = -(g(s_2)+g(t))$, and $g(t)$ are connected, while $g(-(s_1+s_2+t)) = -(g(s_1)+g(s_2)+g(t))$ is not connected. It follows from lemma 5.4 that $[g(s_1)] \cap [g(s_2)] \neq \emptyset$. Since $[s_1] \cap [s_2] = \emptyset$, it is clear from the definition of g that $[g(s_1)]$ and $[g(s_2)]$ both contain the point at infinity, whence s_1 and s_2 are unbounded. □

Recall that, by lemma 5.3, it is legitimate to use the symbol c in $\mathcal{L}(C)$-formulas for purposes of establishing definability. Then we have:

Lemma 5.7 *Let $r \in R$. Then r is bounded if and only if $\mathfrak{R}(C) \models \phi_b[r]$, where $\phi_b(x)$ is the $\mathcal{L}(C)$-formula:*

$$\neg \exists y_1 \exists y_2 (y_1 \leq x \wedge y_2 \leq x \wedge \pi(y_1, y_2) \wedge \neg C(y_1, y_2)).$$

Proof: If r satisfies $\neg \phi_b(x)$ in $\mathfrak{R}(C)$, then, by lemma 5.6, r contains two unbounded regions, so is certainly itself unbounded. Conversely, if r is unbounded, by lemma 4.4, it is simple to construct regions $s_1, s_2 \in R$ such that, $s_1 \leq r$, $s_2 \leq r$ and $[s_1] \cap [s_2] = \emptyset$, and satisfying the $\mathcal{L}(C)$-formula $\pi(y_1, y_2)$. □

Lemma 5.8 *There is no $\mathcal{L}(c,\leq)$-formula ϕ such that, for all $r \in R$, r is bounded if and only if $\mathfrak{R}(c,\leq) \models \phi[r]$.*

Proof: Let $r \in R$ be a Jordan region. We show that r and $-r$ satisfy the same $\mathcal{L}(c,\leq)$-formulas in $\mathfrak{R}(c,\leq)$; the lemma then follows immediately. Let $g : \mathfrak{R} \to \bar{\mathfrak{R}}$ be the model-isomorphism given in lemma 4.7. From the definition of g, it is obvious that both $g(r)$ and $g(-r)$ are Jordan regions in the closed plane. But all Jordan regions in the closed plane are similarly situated, so by lemma 4.6, $g(r)$ and $g(-r)$ satisfy the same $\mathcal{L}(c,\leq)$-formulas in $\bar{\mathfrak{R}}(c,\leq)$. Therefore r and $-r$ satisfy the same $\mathcal{L}(c,\leq)$-formulas in $\mathfrak{R}(c,\leq)$. □

We note in passing the corollary:

Corollary 5.9 *The structures $\mathfrak{R}(C)$ and $\bar{\mathfrak{R}}(C)$ are not elementarily equivalent.*

Proof: By lemma 5.7, the set of bounded regions in R is $\mathcal{L}(C)$-defined in $\mathfrak{R}(C)$ by the formula $\phi_b(x)$. Since there are unbounded regions in R, $\mathfrak{R}(C) \not\models \forall x \phi_b(x)$.
By lemmas 5.1, 5.3 and 5.5 the set $\{(r_1, r_2) \in \bar{R}^2 | [r_1] \cap [r_2] \neq \emptyset\}$ is defined in $\bar{\mathfrak{R}}(C)$ by the formula $\phi_C(x_1, x_2)$ (regarded as an $\mathcal{L}(C)$-formula). Let $r \in \bar{R}$ be nonempty. Then $\bar{\mathfrak{R}}(C) \models \phi_C[r, r]$. Let $s_1, s_2 \in \bar{R}$ be witnesses for the variables y_1 and y_2 in $\phi_C[r, r]$. Then $\bar{\mathfrak{R}}(C) \models \phi_C[s_1, s_2]$ and, thus, $[s_1] \cap [s_2] \neq \emptyset$ and $\bar{\mathfrak{R}}(C) \models C[s_1, s_2]$. Furthermore, $\bar{\mathfrak{R}} \models \pi[s_1, s_2]$. Then, observing the similarity of ϕ_C and $\neg\phi_b$, it is easy to see that $\bar{\mathfrak{R}}(C) \models \phi_b[r]$. The empty region $\emptyset$ clearly satisfies $\phi_b(x)$ in $\bar{\mathfrak{R}}(C)$. Hence, $\bar{\mathfrak{R}}(C) \models \forall x \phi_b(x)$. □

Hence the language $\mathcal{L}(C)$ is expressive enough to distinguish between the open- and the closed-plane polygonal mereotopologies. By lemma 4.7, this is definitely not the case with $\mathcal{L}(c,\leq)$.

5.2 Model-theoretic properties of $\mathfrak{R}$

We briefly review and extend a theorem concerning $\mathfrak{R}(c,\leq)$ proved in [25]. This theorem is crucial for theorems 3.7 and 3.8.

Definition 5.10 *Let T be a theory in the language $\mathcal{L}(\sigma)$. An $\mathcal{L}(\sigma)$-formula $\phi(\bar{x})$ is* complete *in T if, for all $\mathcal{L}(\sigma)$-formulas $\theta(\bar{x})$, exactly one of $T \models \phi \to \theta$ and $T \models \phi \to \neg\theta$ holds. A structure $\mathfrak{A}(\sigma)$ is atomic if every n-tuple in A satisfies a complete $\mathcal{L}(\sigma)$-formula in $Th(\mathfrak{A}(\sigma))$.*

Theorem 5.11 *(Pratt and Lemon [25, theorem 5.11]) $\mathfrak{R}(c,\leq)$ is an atomic model.*

We give a sketch proof of this theorem here, in order to illustrate its relationship to the results derived below. Full details can be found in the given reference.

Sketch proof of theorem 5.11: Let us say that a *connected partition* in R is a set of nonempty, pairwise disjoint, connected elements $r_1, \ldots, r_n$ of R which sum to 1. The relation of being an n-element connected partition in R is visibly $\mathcal{L}(c,\leq)$-definable in $\mathfrak{R}(c,\leq)$; and it is observed in [25, theorem 5.11] that, for a fixed n there are only finitely many n-element connected partitions in R upto the relation of being similarly situated. It follows from lemma 4.6 that every n-element connected partition of the plane must belong to one of a finite number of types in $Th(\mathfrak{R}(c,\leq))$, and hence satisfies a complete formula in $Th(\mathfrak{R}(c,\leq))$. Since every n-tuple in R can be formed by summing the elements of some connected partition, every n-tuple in R satisfies a complete formula in $Th(\mathfrak{R}(c,\leq))$ too. □

In this paper, we require the following easy extension of theorem 5.11.

Corollary 5.12 *$\mathfrak{R}(C)$, $\bar{\mathfrak{R}}(c,\leq)$ and $\bar{\mathfrak{R}}(C)$ are also atomic models.*

Proof: By lemmas 5.1 and 5.3, the relation of being an n-element connected partition in R is $\mathcal{L}(C)$-definable in $\mathfrak{R}(C)$. The proof of atomicity then goes through in exactly the same way for $\mathfrak{R}(C)$ as for $\mathfrak{R}(c, \leq)$. That $\bar{\mathfrak{R}}(c, \leq)$ is atomic follows from theorem 5.11 and lemma 4.7. That $\bar{\mathfrak{R}}(C)$ is atomic follows from the atomicity of $\bar{\mathfrak{R}}(c, \leq)$ and from the interdefinability of primitives guaranteed by lemmas 5.1, 5.3 and 5.5. □

It is worth dwelling briefly on the relationship between corollary 5.12 on the one hand and theorems 3.7 and 3.8 on the other. The former states, roughly, that every n-tuple of polygons satisfies a mereotopological formula which fixes the *definable* relations it belongs to; the latter, by contrast, state that every n-tuple of polygons satisfies a mereotopological formula which fixes the *topological* relations it belongs to. Thus, theorems 3.7 and 3.8 represent a strengthening of corollary 5.12. We now come to the key idea used to effect that strengthening.

5.3 Homeomorphisms from automorphisms

Theorem 5.13 *Let $\mathfrak{A}$ be a mereotopology over a topological space X satisfying the following conditions:*

(i) $\mathfrak{A}$ is a Boolean subalgebra of $RO(X)$;

(ii) if $p, q \in X$ and $p \neq q$ then there are $a, b \in A$ s.t. $p \in a$, $q \in b$ and $a \cap b = \emptyset$ (Hausdorffness);

(iii) if $u \subseteq X$ is a closed set and $p \in X \setminus u$ then there are $a, b \in A$ s.t. $u \subseteq a$, $p \in b$ and $a \cap b = \emptyset$ (T_3-separation);

(iv) if $p \in X$ then there is $a \in A$ s.t. $p \in a$ and $[a]$ is compact (strong local compactness);

(v) the relations $\{(a, b) \in A^2 | [a] \cap [b] \neq \emptyset\}$ and $\{a \in A | [a]$ compact$\}$ are $\mathcal{L}$-definable in $\mathfrak{A}$.

Then, if α is an $\mathfrak{A}$-automorphism, there exists a homeomorphism h of X onto itself such that, for all $a \in A$, $\alpha(a) = h(a) =_{def} \{h(p) | p \in a\}$.

In establishing this result, we adapt a technique of Roeper [28], to reconstruct points as equivalence classes of ultrafilters on $\mathfrak{A}$. By showing that automorphisms of $\mathfrak{A}$ map equivalent ultrafilters to equivalent ultrafilters, we obtain corresponding homeomorphisms of the space X onto itself. For the remainder of this section, we assume that $\mathfrak{A}$ satisfies the conditions of theorem 5.13.

Definition 5.14 *Let U be an ultrafilter on $\mathfrak{A}$. We say that U is a* compact *ultrafilter if U contains some u such that $[u]$ is compact.*

Lemma and Definition 5.15 *Let U be a compact ultrafilter on $\mathfrak{A}$. Then the set $\bigcap\{[u] | u \in U\}$ is a singleton. We denote the member of this set by p_U and say that U* converges *to p_U.*

Proof: This is an adaptation of a standard result ([17, Chapter 1, Exercise 2]). We first show that $\bigcap\{[u] | u \in U\}$ contains at least one point. Choose $u_0 \in U$ such that $[u_0]$ is compact. Then $\bigcap\{[u] | u \in U\} = \emptyset$ implies $\bigcup\{X \setminus [u] | u \in U\} = X$, whence $\{-u | u \in U\}$ covers X and hence $[u_0]$. By compactness of $[u_0]$, let $u_1, \ldots, u_n \in U$ be such that $u_0 \subseteq [u_0] \subseteq -u_1 \cup \ldots \cup -u_n \subseteq -u_1 + \ldots + -u_n$, whence $u_0 \cdot u_1 \cdot \ldots \cdot u_n = \emptyset$ contradicting the fact that U is a proper filter. Next we show that $\bigcap\{[u] | u \in U\}$ contains at most one point. Suppose that $p, q \in \bigcap\{[u] | u \in U\}$ with $p \neq q$. By assumption (Hausdorffness), we can find disjoint $a, b \in A$ such that $p \in a$ and $q \in b$. Hence $p \notin [-a]$ and $q \notin [a]$. Since U is maximal, either a or $-a$ is in U, so that either p or q is not in $\bigcap\{[u] | u \in U\}$. □

Lemma 5.16 *Let U be a compact ultrafilter on $\mathfrak{A}$ converging to p_U. If $p_U \in a \in A$ then $a \in U$. Furthermore, there exists $b \in U$ such that $p_U \in b$ and $[b] \subset a$.*

Proof: Suppose $p_U \in a \in A$. Then $p_U \notin [-a]$. Since U is an ultrafilter converging to p_U, $p_U \in [u]$ for every $u \in U$, so $-a \notin U$, whence $a \in U$. For the second part of the lemma, observe that $X \setminus a$ is closed. By assumption (T_3-separation) there are disjoint $b, b' \in A$ such that $p_U \in b$ and $X \setminus a \subseteq b'$. Thus $[b] \subset a$; and by the first part of the lemma, $b \in U$. □

Definition 5.17 *If U and V are ultrafilters on $\mathfrak{A}$, we say U and V are* equivalent *if $[u] \cap [v] \neq \emptyset$ for all $u \in U, v \in V$.*

Lemma 5.18 *If U and V are compact ultrafilters on $\mathfrak{A}$, then $p_U = p_V$ iff U and V are equivalent.*

Proof: The only-if direction is trivial. For the if-direction suppose that $p_U \neq p_V$. By assumption (Hausdorffness), there exist disjoint $a, b \in A$ such that $p_U \in a$, and $p_V \in b$. By lemma 5.16 $b \in V$ and, furthermore, there exists $u \in U$ such that $[u] \subset a$. Hence, $[u] \cap [b] = \emptyset$ contradicting the equivalence of U and V. $\Box$

Lemma 5.19 *Let α be an $\mathfrak{A}$-automorphism and U and V equivalent compact ultrafilters on $\mathfrak{A}$. Then $\alpha(U)$ and $\alpha(V)$ are equivalent compact ultrafilters on $\mathfrak{A}$.*

Proof: It is straightforward to show that α maps ultrafilters to ultrafilters. The result then follows because the relations $\{a \in A | [a] \text{ compact}\}$ and $\{(a, b) \in A^2 | [a] \cap [b] \neq \emptyset\}$ are, by hypothesis, definable in $\mathfrak{A}$. $\Box$

Lemma 5.20 *Let α be an $\mathfrak{A}$-automorphism, $a \in A$, and U a compact ultrafilter with $p_U \in a$. Then $p_{\alpha(U)} \in \alpha(a)$.*

Proof: By lemma 5.16 $a \in U$, and there exists $b \in U$ such that $p_U \in [b]$ and $[b] \subset a$, so that $[b] \cap [-a] = \emptyset$. By assumption $\{(a, b) \in A^2 | [a] \cap [b] = \emptyset\}$ is $\mathcal{L}$-definable in $\mathfrak{A}$. Then, since α is an automorphism, $[\alpha(b)] \cap [-\alpha(a)] = \emptyset$, i.e. $[\alpha(b)] \subset \alpha(a)$. Since $\alpha(b) \in \alpha(U)$, $p_{\alpha(U)} \in [\alpha(b)] \subset \alpha(a)$. $\Box$

Now we can prove theorem 5.13.

Proof of theorem 5.13: Suppose that α is an $\mathfrak{A}$-automorphism. We define the map h by $h(p_U) = p_{\alpha(U)}$ for a compact ultrafilter U on $\mathfrak{A}$. We show: (i) h is well-defined and 1–1, (ii) both the domain and range of h are the set X, (iii) for all $a \in A$, $\alpha(a) = h(a) =_{def} \{h(p) | p \in a\}$ and $\alpha^{-1}(a) = h^{-1}(a)$, and (iv) h and h^{-1} are continuous.
(i) Let U and V be compact ultrafilters on $\mathfrak{A}$ both converging to p. By lemma 5.19 the automorphism α maps equivalent ultrafilters to equivalent ultrafilters. Hence, h is well defined. Applying the same reasoning to α^{-1}, h is 1–1.
(ii) Let $p \in X$. Then $\{a \in A | p \in a\}$ is a filter on $\mathfrak{A}$ and by assumption (strong local compactness) contains some $a \in A$ with $[a]$ compact. By the prime ideal theorem ([17, Chapter 1, 2.16]) this filter can be extended to a (compact) ultrafilter U on $\mathfrak{A}$. By lemma 5.15, U converges to some point p_U. By the assumption of Hausdorffness of A and lemma 5.16 $p = p_U$. Thus, the domain of h is X. Applying the same reasoning to α^{-1}, the range of h is X.
(iii) Let $p_U \in \alpha(a)$ and U be some compact ultrafilter on $\mathfrak{A}$ converging to p_U. By lemma 5.20 $p_{\alpha^{-1}(U)} \in a$. Hence, $p_U = h(p_{\alpha^{-1}(U)}) \in h(a)$. Conversely, let $p_U \in h(a)$. By the definition of h, $p_{\alpha^{-1}(U)} \in a$ and by lemma 5.20 $p_U \in \alpha(a)$. Hence $\alpha(a) = h(a)$. It follows by (i) and (ii) that $\alpha^{-1}(a) = h^{-1}(a)$.
vi) Let $u \subseteq X$ be an open set. By the T_3-separation condition on A, there is for each point $p \in u$ an open set $a_p \in A$ with $a_p \subseteq u$. Thus the set $\mathcal{U} = \{a_p \in A | p \in u\}$ satisfies $\bigcup \mathcal{U} = u$. Then $h(u) = h(\bigcup \mathcal{U}) = \bigcup_{a \in \mathcal{U}} h(a) = \bigcup_{a \in \mathcal{U}} \alpha(a)$ is a union of open sets and hence is itself an open set. Therefore, h^{-1} is continuous. By substituting h^{-1} and α^{-1} for h and a, respectively, h is continuous. $\Box$

5.4 The main results

We are finally in a position to put all the above lemmas together to derive our main results. Theorem 3.2 follows from lemmas 5.1, 5.3, 5.7 and 5.8. Theorem 3.3 follows from lemmas 5.1, 5.3 and 5.5. Theorem 3.6 follows instantly from lemma 4.6.

In order to prove theorems 3.7 and 3.8, we appeal to some standard results in model theory (see [6]).

Theorem 5.21 *Let $\mathfrak{B}$ be an uncountable model in a countable language and S a countable subset of B. Then $\mathfrak{B}$ has a countable elementary submodel $\mathfrak{A}$ with $S \subseteq A$.*

Theorem 5.22 *Let $\mathfrak{A}$ be a countable, atomic model and let $\bar{a}, \bar{b} \in A^n$ $(n \geq 1)$ satisfy the same formulae in $\mathfrak{A}$. Then there is an automorphism of $\mathfrak{A}$ taking $\bar{a}$ to $\bar{b}$.*

Proof of theorems 3.7 and 3.8: First, we note that in the case $\tilde{\mathfrak{R}} = \bar{\mathfrak{R}}$, it suffices to establish the result for the signature $\langle C \rangle$; the result for the signature $\langle c, \leq \rangle$ then follows by theorem 3.3.

If $\bar{a}$ is any n-tuple in $\tilde{R}$, by corollary 5.12 let $\phi(\bar{x})$ be a complete $\mathcal{L}(C)$-formula satisfied by $\bar{a}$ in $\tilde{\mathfrak{R}}(C)$. To show that ϕ is topologically complete, it suffices to show that, if $\bar{b}$ is an n-tuple satisfying ϕ, then $\bar{a} \sim \bar{b}$.

Let $\bar{b}$ be an n-tuple from $\tilde{R}$ with $\tilde{\mathfrak{R}} \models \phi[b]$, then. Let $\tilde{Q}$ be any countable subset of $\tilde{R}$ satisfying conditions (ii), (iii) and (iv) of theorem 5.13 (with $\tilde{Q}$ substituted for A). It is obvious that such $\tilde{Q}$ exist. By theorem 5.21 there is a countable elementary submodel $\mathfrak{A}$ of $\tilde{\mathfrak{R}}(C)$ containing $\tilde{Q} \cup \bar{a} \cup \bar{b}$. Thus, $\mathfrak{A}$ is countable and atomic, and ϕ is a complete formula in $Th(\mathfrak{A})$ satisfied by both $\bar{a}$ and $\bar{b}$. Then, by theorem 5.22, there exists an automorphism α of $\mathfrak{A}$ such that $\alpha(\bar{a}) = \bar{b}$. We check that the model $\mathfrak{A}$ satisfies all the conditions of theorem 5.13. The first condition is immediate; the conditions (ii), (iii) and (iv) follow because $\tilde{Q} \subseteq A$; for the last condition, the definability of the relation $\{a \in A | [a] \text{ compact}\}$ follows in the open-plane case from lemma 5.7, and in the closed-plane case trivially, since everything falls under it. Then, by theorem 5.13, there is a homeomorphism of $\tilde{R}$ onto itself taking $\bar{a}$ to $\bar{b}$. In other words, $\bar{a} \sim \bar{b}$. $\square$

Finally, we can establish theorem 3.9.

Proof of theorem 3.9: We give the proof for the mereotopology $\mathfrak{R}(C)$. Corresponding remarks apply to $\tilde{\mathfrak{R}}(c, \leq)$ and $\tilde{\mathfrak{R}}(C)$. That all $\mathcal{L}^{\omega_1}(C)$-definable relations in $\mathfrak{R}(C)$ are topological follows using the same proof strategy as for lemma 4.6; the details are routine. Conversely, if $\mathcal{C}$ is a topological relation over R, then

$$\bigvee \left\{ \phi(\bar{x}) \in \mathcal{L}(C) \;\middle|\; \begin{array}{l} \phi \text{ is a topologically complete } \mathcal{L}(C)\text{-formula} \\ \text{s.t. } \mathfrak{R}(C) \models \phi[\bar{a}] \text{ for some } \bar{a} \in \mathcal{C} \end{array} \right\}$$

is a formula of $\mathcal{L}^{\omega_1}(C)$ (by the countability of $\mathcal{L}(C)$), and is clearly satisfied in $\mathfrak{R}(C)$ by all and only those n-tuples in $\mathcal{C}$. $\square$

6 Related Work

So far, relatively little effort has been devoted to the investigation of the expressive power of specific mereotopological languages. We mentioned [14] as an exception. However, formal languages and their expressive power have been investigated in the more general setting of topology and geometry. Several modal languages have been employed to capture spatial notions (e.g. [27, 1, 2, 10]) with varying degrees of success (see [18]). First-order languages have been investigated in the context of o-minimal structures where variables are interpreted over elements of a dense linear order (e.g. [16, 23, 22, 30]). The expressive power of these languages is restricted such that the definable sets form well-behaved subsets of topological spaces. Bankston developed in [3] the notion of a first-order representation which maps topological spaces to $\mathcal{L}$-structures such that homeomorphic spaces get mapped to isomorphic structures. This notion allows us to compare the expressive power of first-order languages with respect to classes of topological spaces. This work is extended in [4]. In order to reference all entities of a topological space, i.e. points *and* sets, a (monadic) second-order language $\mathcal{L}_t$ was studied in [12, 32].

7 Conclusion

In this paper, the expressive power of the mereotopological languages $\mathcal{L}(c, \leq)$ and $\mathcal{L}(C)$ over the domain of regular polygons in the open and closed planes has been investigated. It was shown that, in the open plane, $\mathcal{L}(C)$ is strictly more expressive than $\mathcal{L}(c, \leq)$. In fact, $\mathcal{L}(C)$ is adequate to characterize every n-tuple of polygons in the open plane upto topological similarity. It was further shown that, in the closed plane, the two languages are equally expressive, and suffice to characterize any n-tuple of polygons in the closed plane upto topological similarity. These results were then used to prove a theorem about the topological adequacy of the infinitary versions of these languages. One additional corollary was that the language $\mathcal{L}(C)$ contains a sentence distinguishing the open plane from the closed plane, while the language $\mathcal{L}(c, \leq)$ does not.

Although we have restricted attention to the polygonal domains, only certain aspects of these domains' well-behavedness were appealed to: specifically, the atomicity of the relevant $\mathcal{L}(c, \leq)$- and $\mathcal{L}(C)$-structures, and the accessibility lemma 4.4. Other, more liberal, domains of quantification also possess these properties, most saliently, the domain of regular definable sets in the open or closed planes. (See [25], section 6 for a discussion.) Thus, the definability results reported here will carry over unproblematically to these richer domains. However, we at present lack a general characterization of just how far the polygonal mereotopologies can be liberalized without affecting their model-theoretic properties. The investigation of topologically adequate languages for corresponding mereotopologies in three dimensions remains to be systematically investigated.

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