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Developing a theory

To study these problems, we need:

1. a model of fitness landscape containing some features of realistic problems
2. A set of variables with which to describe the behavior
3. A method of replacing the stochastic dynamics with deterministic dynamics.

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Overview of our approach

1. Functions of an additive trait.

   Trait: $h = \sum_{i=1}^{L} J_i x_i$

   Objective function: (possibly probabilistic) function of weighted sum of allele values,

   $$E = \mathcal{E}(h) + noise$$

   where J_i are fixed and determine the importance of each site.

2. Macroscopic variables

- Distribution of the trait in the population at time t, $\boxed{\rho_t(h)}$.
- Parametrized by a cumulant expansion,

Cumulants

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$$\begin{aligned}
\kappa_1 &= \text{mean}\\
\kappa_2 &= \text{variance}\\
\kappa_3 &= \text{related to "skewness"}\\
\kappa_4 &= \text{related to "kurtosis" or "excess"}\\
&\vdots\\
\kappa_n &= \text{measure of } n\text{th moment relative to a Gaussian}
\end{aligned}$$

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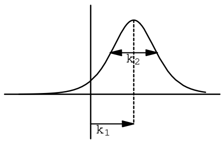
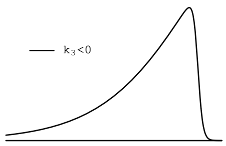
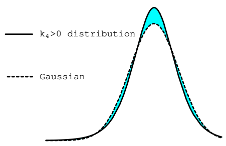
First two cumulants:



A distribution with negative "skewness":



A distribution with positive kurtosis:



Cumulants average well. They contain all information about the distribution.

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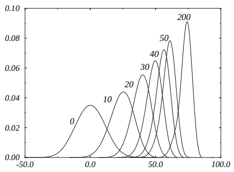
3. Averaged over all realizations of the stochasticity (ensemble average)

$$\langle \rho_t(h) \rangle_e \rightarrow \langle \rho_{t+1}(h) \rangle_e$$



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Example — change in mean due to selection

$$\kappa_1 = \sum_\mu h^\mu$$

$$\Downarrow$$

$$\kappa_1^{(s)} = \sum_\mu h^\mu \boxed{n^\mu} \quad \longleftarrow$$ Average over number of times μ selected n^μ.

$$\Downarrow$$

$$\sum_\mu \boxed{h^\mu} \frac{W^\mu}{\sum_\nu W^\nu} \quad \longleftarrow$$ Average over ways of producing $\{h^\mu\}$

$$\Downarrow$$

$$\left\langle \kappa_1^{(s)} \right\rangle_e = \boxed{\boxed{\left\langle \frac{\sum_\mu h^\mu W^\mu}{\sum_\mu W^\mu} \right\rangle}} \quad \longleftarrow \quad \langle \cdot \rangle = \prod_\mu^N \int dh^\mu \, \langle \rho_t(h^\mu) \rangle_e \, .$$

A complex multi-dimensional integral.

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Expanding around the infinite population solution

In an infinite population: the situation is simple,

$$\left\langle \kappa_1^{(s)} \right\rangle_e = \left\langle \frac{\sum_\mu E^\mu W^\mu}{\sum_\mu W^\mu} \right\rangle = \frac{\overline{EW}}{\overline{W}}$$

where $\bar{\cdot}$ denotes averaging over the distribution $\langle \rho_t(E) \rangle_e$.

An expansion for weak selection, large N: (A non-rigorous method):
Write the expression as

$$\left\langle \kappa_1^{(s)} \right\rangle_e = \left\langle \frac{\sum_\mu E^\mu W^\mu}{N\overline{W} + \sum_\mu \left(W^\mu - \overline{W}\right)} \right\rangle$$

and expand in small $\sum_\mu \left(W^\mu - \overline{W}\right)$

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$$\left\langle \kappa_1^{(s)} \right\rangle_e = \left\langle \frac{\sum_\mu E^\mu W^\mu}{N\overline{W}} \left[1 - \frac{\sum_\mu \left(W^\mu - \overline{W}\right)}{N\overline{W}} + \frac{\sum_\mu \left(W^\mu - \overline{W}\right)^2}{\left(N\overline{W}\right)^2} - \dots \right] \right\rangle .$$

Example: let $W = 1 + sh$ with selection strength s. Then

$$\kappa_1^{(s)} = \kappa_1 + s\left(1 - \frac{1}{N}\right)\kappa_2$$

Carrying on . . . ,

$$\begin{aligned}
\kappa_1^{(s)} &= \kappa_1 + s\left(1-\frac{1}{N}\right)\kappa_2,\\
\kappa_2^{(s)} &= \left(1-\frac{1}{N}\right)\kappa_2 + s\left(1-\frac{3}{N}+\frac{2}{N^2}\right)\kappa_3,\\
\kappa_3^{(s)} &= \left(1-\frac{3}{N}+\frac{2}{N^2}\right)\kappa_3 + s\left(1-\frac{7}{N}+\frac{12}{N^2}-\frac{6}{N^3}\right)\kappa_4\\
&\quad - s\frac{6}{N}\left(1-\frac{1}{N}\right)\kappa_2^2,\\
\kappa_4^{(s)} &= \left(1-\frac{7}{N}+\frac{12}{N^2}-\frac{6}{N^3}\right)\kappa_4 - \frac{6}{N}\left(1-\frac{1}{N}\right)\kappa_2^2\\
&\quad + s\left(1-\frac{15}{N}+\frac{50}{N^2}-\frac{60}{N^3}\right)\kappa_5 - \frac{60s}{N}\left(1-\frac{3}{N}+\frac{6}{N^2}\right)\kappa_2\kappa_3
\end{aligned}$$

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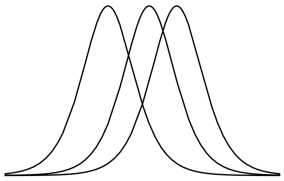
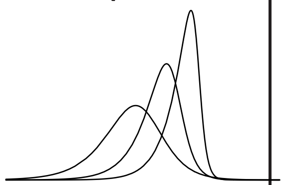
Punch lines about selection and cumulants

- Finite population introduces higher cumulants through *non-linear* interactions.
- The stronger the selection pressure, the more important these effects are.

Infinite Population:



Finite Population:



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Aside — Expanding around the ensemble average

(Prügel-Bennett, 1997.) What to do when fluctuations around the ensemble average are important?

Use further macroscopics — the variation in $\langle \rho_t(h) \rangle_e$ as parametrized via the co-variances

$$v_{n,m} = \langle \langle \kappa_n \kappa_m \rangle_e \rangle_e - \langle \langle \kappa_n \rangle_e \rangle_e \langle \langle \kappa_m \rangle_e \rangle_e$$

where $\langle \langle \cdot \rangle_e \rangle_e$ denotes averaging over the assumed multi-variant Gaussian form of the ensemble fluctuations.