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Cumulant dynamics in a simple Wright-Fisher model

Rattray and Shapiro, 1999, http://www.cs.man.ac.uk/ai/Papers/magnus/mloci_v2.ps.gz

The model:

Trait: $t^\alpha = \sum_{i=1}^{L} x_i^\alpha$

Selection: Simple multiplicative landscape,

$$W_\alpha = \exp s \sum_i x_i^\alpha; \quad p_\alpha = \frac{W_\alpha}{\sum_{\alpha'} W_{\alpha'}}$$

Recombination: Assume that this happens many times prior to each selection, so that it is at the linkage equilibrium fixed point.

$$P(x_1, x_2, \ldots, x_L) = \prod_i p(x_i) = \prod_i \tau_i^{x_i} (1 - \tau_i)^{(1-x_i)}$$

where τ_i is the allele frequency at site i, $\tau_i = N^{-1} \sum_\alpha x_i^\alpha$.

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Mutation: $x_i^\alpha \to 1 - x_i^\alpha$ with probability u.

Expand around the infinite population solution

e.g.

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$$\begin{aligned}\langle \kappa_1 \rangle_e &= \sum_i \frac{\tau_i e^s}{\tau_i e^s + 1 - \tau_i} \\ &\quad - \frac{1}{N} \sum_i \frac{\tau_i e^{2s}}{(\tau_i e^s + 1 - \tau_i)^2} \exp\left(L f(\vec{x})\right)\end{aligned}$$

Only in an expansion in small s can this be expressed in terms of the cumulants before selection.

Expanding to first order in s and N

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$$\begin{aligned}\left\langle \kappa_1^{(s)} \right\rangle_e &= \kappa_1 + s\kappa_2, \\ \left\langle \kappa_2^{(s)} \right\rangle_e &= \left(1 - \frac{1}{N}\right)\kappa_2 + s\kappa_3, \\ \left\langle \kappa_3^{(s)} \right\rangle_e &= \left(1 - \frac{3}{N}\right)\kappa_3 + s\kappa_4 \\ \left\langle \kappa_4^{(s)} \right\rangle_e &= \left(1 - \frac{6}{N}\right)\kappa_4 - \frac{1}{N}\kappa_2 + s\kappa_5 \\ &\vdots \qquad \vdots\end{aligned}$$

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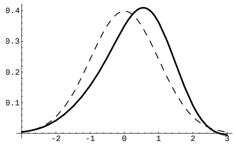
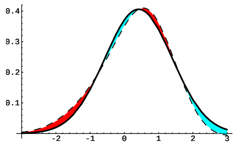
Selection and Crossover Effects

Selection increases the mean, reduces the variance, and introduces skewness



Crossover restores higher cumulants such as skewness to random values, mediated by the degree of correlation in the population.



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Mutation

Likewise, the cumulants after mutation can be expanded out,

$$\begin{aligned}
\left\langle \kappa_1^{(m)} \right\rangle_e &= \kappa_1 (1-2u) + Lu, \\
\left\langle \kappa_2^{(m)} \right\rangle_e &= \kappa_2 (1-4u) + Lu, \\
\left\langle \kappa_3^{(m)} \right\rangle_e &= -2u\kappa_1 + (1-6u)\,\kappa_3 + Lu, \\
\left\langle \kappa_4^{(m)} \right\rangle_e &= -8u + (1-8u)\,\kappa_4 + Lu, \\
\vdots \quad & \quad \vdots
\end{aligned}$$

The diffusion limit

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Taking, $N \to \infty, \delta t \equiv N^{-1}, S \equiv Ns$, and $U \equiv Nu$,

$$\frac{\mathrm{d}\mathbf{k}}{\mathrm{d}t} = -\mathbf{G}(U,S)\mathbf{k} + UL\,\mathbf{b}\ ,$$

where $\mathbf{k} = (\langle\kappa_1\rangle_e, \langle\kappa_2\rangle_e, \langle\kappa_3\rangle_e, \ldots)^T$, $\mathbf{b} = (1,1,\ldots,1)^T$ and,

$$\mathbf{G}(U,S) = \begin{pmatrix} 2U & -S & 0 & 0 & \cdots \\ 0 & 1+4U & -S & 0 & \cdots \\ 2U & 0 & 3(1+2U) & -S & \cdots \\ 0 & (1+8U) & 0 & 2(3+4U) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\ .$$

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- A wholly linear system (albeit in infinite dimensions).
- L is a trivial scaling factor.

Solution:

$$\mathbf{k}(t) = \mathbf{k}^* + e^{-\mathbf{G}t}\left[\mathbf{k}(0) - \mathbf{k}^*\right]\ .$$

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Adiabatic elimination

For small U and S,

- There is a single slow mode,
- Most of the slow mode is in κ_1
- The higher cumulants evolve quickly to "fixed points" which vary slowly only through κ_1.

(True for small U to 2nd order in S in perturbation expansion.)

$$\frac{\mathrm{d}\kappa_1}{\mathrm{d}t} \simeq S\kappa_2^* + U(L - 2\kappa_1)$$

where, to four cumulants,

$$\kappa_2^* \simeq \frac{UL(18 + S^2 + 60U + 48U^2 - 2S(3 + 4U)(2\kappa_1/L - 1))}{S^2(1 + 8U) + 6(3 + 22U + 48U^2 + 32U^3)}$$

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Modeling a changing environment

$$h^\alpha = \sum_{i=1}^{L} (J_i x_i^\alpha + (1 - J_i)(1 - x_i^\alpha))\,;\ \text{where } J_i \in \{0, 1\}\ .$$

$$J_i \to (1-\epsilon_i)J_i + \epsilon_i(1-J_i) \quad \text{where} \quad \epsilon_i = \begin{cases} 1 & \text{with probability } \theta\ , \\ 0 & \text{with probability } 1 - \theta\ . \end{cases}$$

Like mutation, except each site experiences the same change. Thus,

$$u^n \to \langle \epsilon_i^n \rangle_e = \theta,$$

in the mutation results. Since all moments are of the same order, it is hard to see how this could be approximated using a diffusion equation.

Optimal mutation rate in diffusion limit ($\theta = N\Theta$)

$$U_{\text{opt}} \sim \tfrac{1}{2}\sqrt{\Theta} \quad \text{as } S \to 0\ .$$

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Expanding around the linkage equilibrium solution

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