Machine-Efficient Chebyshev Approximation for Standard Statistical Distribution Functions

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Abstract

When implementing a real function such as sin, cos, etc. on any computing system, polynomial approximations are almost always used to replace the actual function. The reason behind this is that addition, subtraction, and multiplication are efficiently implemented in general-purpose processors [2].

In this paper we extend the idea of Brisebarre, Muller, Tisserand, and Chevillard on machine-efficient Chebyshev approximation [2,3]. Our extensions include standard statistical distribution functions, by which we mean: the normal distribution, the beta distribution, the F-distribution, and the Student's t-distribution. We choose Chebyshev polynomials as they provide a good polynomial approximation [1,5]. These practical calculations have been performed using Müller's iRRAM [6].

The standard statistical distribution functions are important functions that are used in probability theory and statistics [4]. Finding an efficient way of approximating these functions would be beneficial. The (cumulative) normal distribution function N(0,1) with mean 0 is defined by the following Equation.

$$N_{0,1}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}u^2}\, du \tag{1}$$

On the other hand, the beta distribution function is defined in Equation 2. It is defined in the interval [0,1] with two positive shape values α and β.

$$F(x;\alpha,\beta) = \frac{B_x(\alpha,\beta)}{B(\alpha,\beta)} = I_x(\alpha,\beta) \tag{2}$$

where $B_x(\alpha,\beta)$ is the incomplete beta function, $B(\alpha,\beta)$ is the beta function, and $I_x(\alpha,\beta)$ is the normalized incomplete beta function [4].

The F-distribution function is defined in Equation 3. It is defined in the interval [0,∞] with the degree of freedom values v_1 and v_2.

$$Q(F|v_1, v_2) = I_x(\frac{v_1}{2}, \frac{v_2}{2}) \text{ , where } x = \frac{v_1 F}{v_1 F + v_2} \text{ and } F \geq 0, v_i > 0 \quad (3)$$

The Student's t-distribution is a special case of the F-distribution and is defined in Equation 4.

$$A(t|v) = 1 - I_x(\frac{v}{2}, \frac{1}{2}) \text{ , where } x = \frac{v}{v + t^2} \text{ and } v > 0 \quad (4)$$

The first step of getting the machine-efficient Chebyshev approximation is to calculate the Chebyshev series that approximate the required functions. The next step is to convert the standard approximation into a machine-efficient version. We implement the machine-efficient Chebyshev approximation as shown in Equation 5.

$$f(x) \approx \sum_{n=0}^{\infty} \frac{a_n}{2^m} x^n \text{ where } a_0, ..., a_n \in \mathbb{Z}, \text{ and } m \in \mathbb{N} \quad (5)$$

The machine-efficient Chebyshev approximation can approximate the above functions with a defined error. We found that these machine efficient approximations do indeed improve the efficiency with which these operations can be performed. The total error(E_t) of performing the machine-efficient approximation is defined in Equation 6.

$$E_t = \text{machine-efficient error} + \text{approximation error} \quad (6)$$

The machine-efficient error that is caused by converting the standard Chebyshev approximation to machine-efficient Chebyshev approximation. This error is defined in Equation 7.

$$\text{Machine-Efficient error} = n \cdot \frac{1}{2^m} \quad (7)$$

where n is the order of Chebyshev approximation, and $m \in \mathbb{N}$ that represents the accuracy of the coefficients.

The approximation error is defined in Equation 8.

$$|f(x) - P_n(x)| \leq \frac{2(b-a)^{n+1}}{4^{n+1}(n+1)!} \, max_{a \leq x \leq b} |f^{n+1}(x)| \quad (8)$$

where a, and b are the end points of the interval [a,b], and n is the order of interpolating polynomial $P_n(x)$ [1].

Instead of doing forward error analysis, Equation 6 can be used to perform this type of approximation in an exact arithmetic framework.

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